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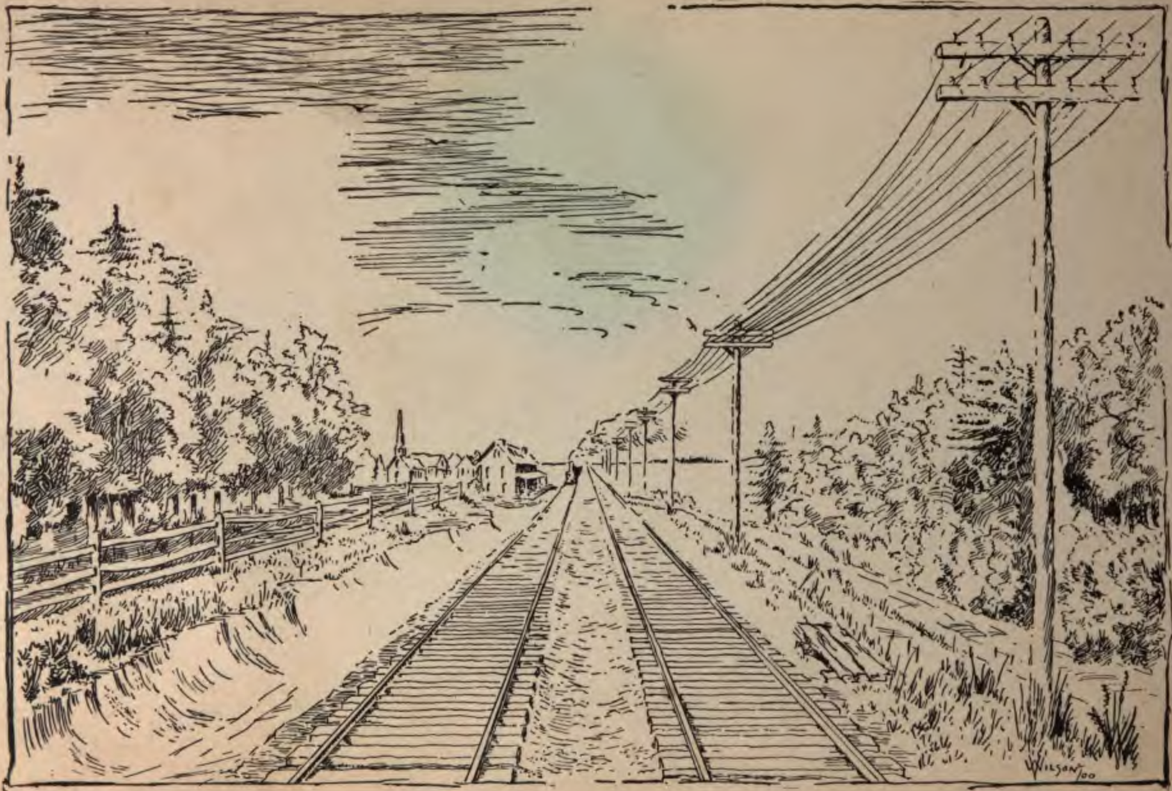

THE FAMILIAR RAILROAD-TRACK. (See page 2.)

FREE-HAND PERSPECTIVE.

FOR USE IN MANUAL-TRAINING
SCHOOLS AND COLLEGES.

BY

VICTOR T. WILSON,
PROFESSOR OF ENGINEERING DRAWING,
PENNSYLVANIA STATE COLLEGE.

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PREFACE.

IN presenting a new treatise on an old theme, it seems necessary at the outset to give a brief history of its development, so that the reader may see more clearly its purpose. When the author began to teach the elements of freehand drawing to technical students, and found it necessary to deal with the art of drawing associated with the principles of linear perspective, he was impressed with the fact that there was an unnecessary and undesirable antagonism between the two, which, when speaking of the one, required the other to be ignored, at the same time both had to be carried along side by side; the time seemed very long before the student might be permitted to view the mystery of why the one was dependent upon the other, in fact the time was so long that the connecting of the two in a practical manner had to be accomplished after the student left the teacher.

Owing to this apparent discord, an effort was made to connect into a harmonious result these two subjects hitherto treated separately. It was not an easy task, and the endeavor, coupled with the necessary research, extended over a period of several years. It is not intended to claim that the task has been completed, if indeed it be capable of complete treatment, the one phase of the subject blending into the other, as it does, by indistinguishable gradations, but the author feels that he has reached a point where the fruits of his labors are at least ripe enough to share with others.

The subject, essentially in its present form, has been taught to students in mechanical engineering, though to them rather as a class of men whose needs are identical with a larger general class for whom higher education means, among other things, general proficiency in the art of sketching, so that graphic expression takes equal place with verbal or written expression. Of the classes mentioned, the technical students have perhaps less inclination towards artistic expression, for with them whatever ability is acquired in sketching is much more readily appropriated and digested when given in the form of well-defined principles, which enable them to draw by the aid, mainly, of careful analysis with which they are familiar, and which is a chief requisite of proficiency in the sciences.

The discussion of the principles of linear perspective as applied to free-hand sketching is, then, the subject of this book. The discovery of the principles themselves is not of recent origin; as a science, conical projection has long been formulated, but its application to sketching has been strangely neglected.

The consecutive treatment observed by others is in the main adhered to, but the liberty has been taken of abbreviating that of certain divisions of the subject to a degree which may seem almost heresy to those accustomed to traditional methods; these parts have been regarded as of minor importance in using free sketch methods. For example, in a large share of the problems to be given the student to work, it is understood that vanishing-points and measuring-points are to be dispensed with; dependence upon geometrical construction being considered an obstacle to free artistic expression. Several features, on the other hand, have been given fuller treatment than is usually accorded them, such as the estimation of the proper convergence of parallel lines, the approximate values in varying positions, of limited plane surfaces, etc. One department is introduced which, it is thought, is entirely new in its application; this is the use of the directed plane formed by the rotation of a definite line, to furnish a key for estimating approximately the perspective values of plane angles. This device forms a connecting link, hitherto ignored, which makes it possible to approximate all of the data necessary to represent simple forms perspectively under any given conditions. It is the purpose, therefore, by the treatment adopted, so to equip the student that he may give graphic expression to his conceptions of simple forms, to draw from description, from imperfect data, or lastly, what is particularly useful to those who are to deal with the crafts, to sketch perspectively from the working drawing so that they may give in turn to the uninitiated, to whom the conventional drawings may be unintelligible, a clear idea of forms.

In order to make the connection complete between the study of form by lines and by shadow, it has seemed necessary to trespass upon the field of descriptive geometry, as commonly understood, for an analysis of certain conditions, that is, the study of tangent and intersecting planes, as they are at the basis of the delineation of shadow effects. In so doing it is desired not to make the subject more abstruse, but to add clearness to those constructive processes which have been found difficult to comprehend adequately when presented in orthographic projection.

No effort is made towards an exhaustive treatment of shadow phenomena, as there are grave difficulties in the way of any attempt to empiricise light effects; only those few well-understood principles that have long been known are dwelt upon at any length.

In closing, the author wishes to acknowledge indebtedness to his former teacher, Professor Leslie W. Miller, for the inspiration which has led to this work, and also for the additional help gained from the latter's book on "The Essentials of Perspective," which, for the purposes of the artist, stands almost if not quite alone in its field at the present time.

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FREE-HAND PERSPECTIVE.

CHAPTER I.

INTRODUCTION.

1. **Definition of Perspective.**—Perspective is the science of representing form pictorially by fixing the exact location, under certain conditions, of lines which represent the contours of objects or the separation of surfaces. It is an exact science and is a division of the subject of projective or descriptive geometry. It is sometimes called conical projection as well.

It has, however, a broader meaning when it is made to include aerial effects, delicate differences of color and light and shade, or, in other words, those things which tend to make a picture more real and lifelike. The term is often used in this last sense, although there is no recognized connection between the scientific perspective and the artist's perspective, for a mastery of this last is only obtained by a long and exhaustive study of nature, a study by sympathetic observation unhampered by rules.

2. **A Picture, and How it may be Made.**—If you look out of a window you will see before you a certain amount of that which is beyond; stepping closer to the window, it will be possible to see more within its limits; moving further away, less is seen. Now if in any one position the window can be reached with a marking-point and the glass sized with glue, it is possible to follow with lines the edges of the objects

beyond it, a rough approximation being obtained to the appearance of those objects. So that a picture may be described as a drawing which, by its lines and arrangements of light and shade, etc., follows fairly closely the forms of objects as viewed through a window, giving a clear impression of what they are. Pictures are not made practically in this way, of course, but it serves to illustrate the meaning.

It may be observed, as a corollary to the above, that the lines in any such picture would lie directly over the edges of the objects were it possible to realize the conditions under which the picture of them was made, that is, the conditions as to the position of the observer and that of the imaginary pane of glass.

3. Familiar Phenomena of Perspective.—*Receding parallel lines seem to converge towards a common vanishing-point.* If you stand on the sidewalk of a street of city houses, for example, built up in long rows of similar type, and look along that street, the receding edges of these forms, edges formed by the roof-cornices, the sidewalk, etc., and imaginary lines through the lintels of the windows if they be of the same size and similarly situated, do not look actually parallel as we know them to be, but seem to converge to a common point. If this is not at first apparent, it can be made so by holding up a pencil or other straight edge, and moving it in a vertical plane, perpendicular to the line of observation until it lies over the different edges in turn. It will immediately be seen that the pencil is constrained to move radially about a point which is directly in front of and on a level with the eye.

Another familiar illustration may be found by looking along a straight railway track, the more level the better. The rails are more easily followed towards their apparent meeting-point than are the edges along the street. If fairly regular telegraph-poles line the track also, imaginary lines through the tops and bases will be seen, by the use of the pencil as just described, to head towards a common point. And, if the country is level and open, it will also be seen very much more clearly that this point of apparent intersection is on the horizon in front of the eye.

Additional illustrations could easily be obtained, and, by the use of the pencil, the truth observed can be extended to apply to any system of parallel lines no matter what their direction.

4. A Vanishing-point.—'Vanishing-point' is the term usually adopted to designate the picture of that point towards which parallel lines seem to converge. For example, as compared with the dimensions of an ordinary drawing, the point in which the car-rails seem to meet on the horizon is at an infinite distance, hence the meaning of the term vanishing-point in the picture.

5. Distances of Objects Determine their Apparent Size.—*The further an object recedes from the observer the smaller it becomes*, or, in other words, the smaller is the visual angle which it subtends.

Standing on a railway track and looking along it while a train approaches in the distance, it will be noticed that it first appears as a mere speck on the horizon and increases in size as it draws nearer. In fact it is quite a common and familiar experience to guess at the speed of an approaching train by the rapidity with which it seems to increase in size. Again, looking at the spaces between the telegraph-poles or railroad-ties, it is noticed that the nearer spaces appear the larger. These changes in size due to the difference in the distances of objects can be verified by numerous every-day examples: people walking along the streets, moving vehicles, boats sailing on the water, near and distant trees, buildings, etc.

The observations just recorded of perspective phenomena are fundamental to the subject, and should be carefully noted, and any additions to them made that can be readily thought of. Sketch-work depends directly upon an observance of the laws underlying these phenomena.

6. Transparent-plane Illustration. — Since perspective rules are to be used as an aid in making offhand sketches, and since, therefore, the pictorially graphic element is to predominate, it will be well first to take up the picture on a transparent plane as it might be made by an artist and consider the conditions applying to the phenomena just observed.

Fig. 1 shows the picture of a transparent plane or pane of glass upon which an artist has drawn a block as he sees it, lying on the ground on the other side of the glass. He is supposed to stand with head very still so that his eye can be regarded as a point. The dotted lines show construction that will be referred to more in detail in a later example.

It can be seen that if lines are drawn from the different angles of the object to the artist's eye these lines will cut the transparent plane in points, which, from what was stated in section 2, will be the pictures of points of the object; in fact the artist marks them by putting his pencil or brush over the separate points or angles in turn. If two neighboring points on the transparent plane be joined by a line, as shown, that line will, from the position of the observer, seem to lie directly over an edge of the object beyond; two neighboring parallel lines thus drawn are seen to converge towards some point in the same plane, and by prolonging the lines to their intersection this can be found. It will be the picture of the apparent meeting-point of the actual lines. This may be seen to be the case by reference to the construction in the picture. *AB* is the edge of the block nearest to the artist's eye; *ab* is the picture of that edge obtained by the imaginary converging lines from *AB* to the observer's eye. If the edge of the block just under and parallel to *AB* is prolonged, the apparent meeting-point of the two is found to be on the actual horizon. If *ab* and the lower parallel edge on the transparent plane are similarly prolonged, *c* is found to be the picture of their apparent meeting- or vanishing-point. *C* is now seen to lie upon a straight line from *c* to the artist's eye. Taking then the two points *C* and *c* together, it may be said that *c* is the picture of that point of either line which is infinitely distant.

It is also instructive to note, since the line *ab* prolonged

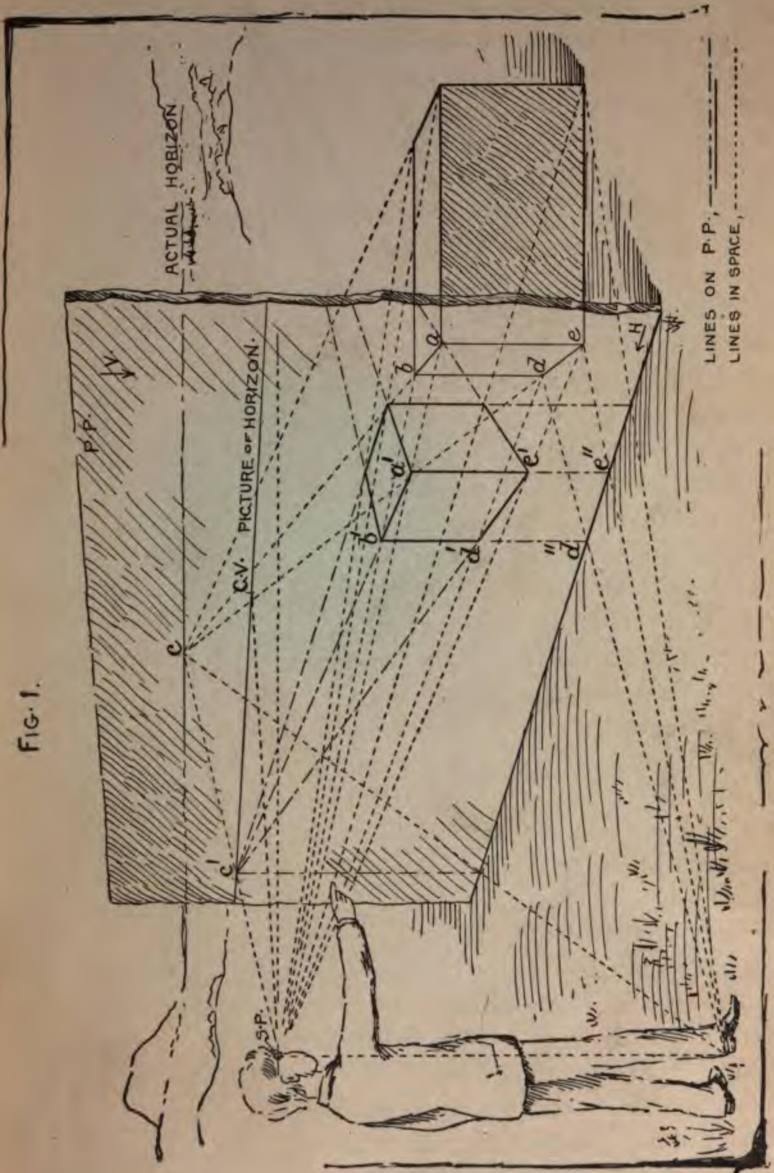
ACTUAL HORIZON
P.P.
V
PICTURE OF HORIZON
C.V.
c
c'
a
b
d
e
a'
b'
d'
e'
d''
e''
H
S.P.
LINES ON P.P.,
LINES IN SPACE,

FIG. 1.

vanishes at *c*, and the line *S.P.c′* also goes through *c*, that *c′* can be located on the transparent plane by imagining a line as drawn from *S.P.* actually parallel to *ab*; in other words, by looking in the direction of the line *ab*. From this can be deduced a rule for finding the vanishing-point of any line, which may be stated as follows:

7. Rule for Finding the Vanishing-point of a Line.—To find the vanishing-point of a line draw an imaginary line of direction parallel to the line itself, and where this line of direction intersects the plane of the picture will be the vanishing-point required.*

How, it may be asked, could this be applied to include the vanishing-point of the vertical edges of the block? If a line of direction is taken by looking upward or downward, parallel to these edges, and to the transparent plane, it will meet the plane at infinity, and hence the vanishing-point of the vertical edges is infinitely distant in a direction parallel to the picture-plane. The same would be true were the edges parallel to the picture-plane in any other position.

8. Demonstration by the Transparent Plane that Objects Grow Smaller as they Grow Distant.—Consider the superimposed triangular planes formed by the lines between *S.P.* and *ab* and *a′b′*. Since *a′b′* is the picture of the edge *ab*, being in fact the base of a similar triangle having *ab* as *its* base, it can readily be seen that if the block be assumed to move away from the observer along a line perpendicular to the transparent plane, then since the base *ab* remains the same length, the altitude of the triangle *S.P.ab* will increase, that of *S.P.a′b′* remaining fixed. Hence the side *a′b′* parallel to the base will grow smaller. This observation as to change in size can be extended to apply to a solid as well as a triangle; for example, to the square pyramid formed by the lines from *S.P.* to the points *a*, *b*, *d*, *e*, and the plane parallel to its base *a′*, *b′*, *d′*, *e′*, in fact to any group of lines from the object to the point *S.P.*

---

* This will have fuller discussion in Chapter III.

9. General Relation of Observer, Transparent Plane, and Object.—From the above observations it can be readily seen that if, instead of the object being moved further away, the transparent plane had been conceived to move nearer the object, then the picture would become larger, for, referring to the triangles again: Of lines drawn parallel to the base of a triangle, those nearer to the base will be larger than those further away. Again, if the object and transparent plane were to remain fixed and the observer moved further away, then obviously the picture on the transparent plane would become larger because, as the altitude of a triangle is increased, a line drawn parallel to the base at a fixed distance from it will become larger.* Now it can be seen from these observations that for a given position of the observer, tranparent plane, and object, but one picture can result. If either or all distances are changed, the picture will change.†

As a corollary to this, and easily verified by reference to the figure, or to any similar observation of every-day experience, if the eye be moved to the right or left, or up or down, or in any way changed with reference to the transparent plane, then the picture will change accordingly. In Fig. 1, for example, we can see that the effect of raising the eye or point *S.P.* would undoubtedly be to make visible more of the top of the block, and of lowering it would be to see less. Fig. 2 is introduced to make this more clear. The artist has moved slightly to his right, nearer to the reader, as well as depressed his head somewhat.

10. Arbitrary Divisions of Perspective.—Arbitrary division of the subject of perspective is sometimes made according to the number of vanishing-points which are involved in the drawing, that is, one-, two-, or three-point perspective; the first is sometimes called parallel perspective because, in this case, the principal edges of a rectangular solid are either parallel or perpendicular to the plane of the picture.

---

* If the observer is assumed to move to an infinite distance, the lines of direction from observer to object will become parallel, and then the conditions are those of ordinary orthographic projection or descriptive geometry.

† There is no limit, theoretically, to the distance from *S.P.* at which the transparent plane can be placed; practically, however, it is invariably taken for convenience either tangent to the nearest corner of the object or is placed somewhere between the *S.P.* and the object, as shown in Fig. 1.

If, in Fig. 1 or 2, the block be imagined as turned so that its nearest vertical face is parallel to the transparent plane, an example of 'one-point perspective' will be had, and the following facts can be noted: the four edges of the face would be parallel to the transparent plane, and the vanishing-point of the two horizontal edges, which before was at *c'*, would have moved to infinity towards the artist's left; the horizontal receding edges would be perpendicular to the transparent plane, hence actually parallel to the imaginary line *S.P. C.V.*, and have *C.V.* as their vanishing-point. Only this one vanishing-point, then, figures in the drawing, hence the term. The position of the object shown in Fig. 1 gives an example of 'two-point perspective' because there are two vanishing-points to consider, those of the receding edges towards the right and those towards the left. If now the block were imagined to be tilted up on one corner, it can be readily seen that, since no edge would be parallel to the transparent plane, there would be three vanishing-points to consider, hence the name given to this would be 'three-point perspective.'

In what follows no direct reference will be made to either one of the three divisions because, in sketching, it is of no practical importance; it is mentioned here merely as an explanation of terms in quite common use.

11. Perspective Notation.—For convenience in demonstration and referring to figures it is necessary to note a few brief definitions and explanatory notation; few will be used, however, as they are but secondary to the work in hand and only requisite as stepping-stones to a more general grasp of the subject.

The Station-point, or S.P. as it is generally defined, sometimes also called 'point of sight,' is the position of the artist's eye. See Figs. 1 and 2.

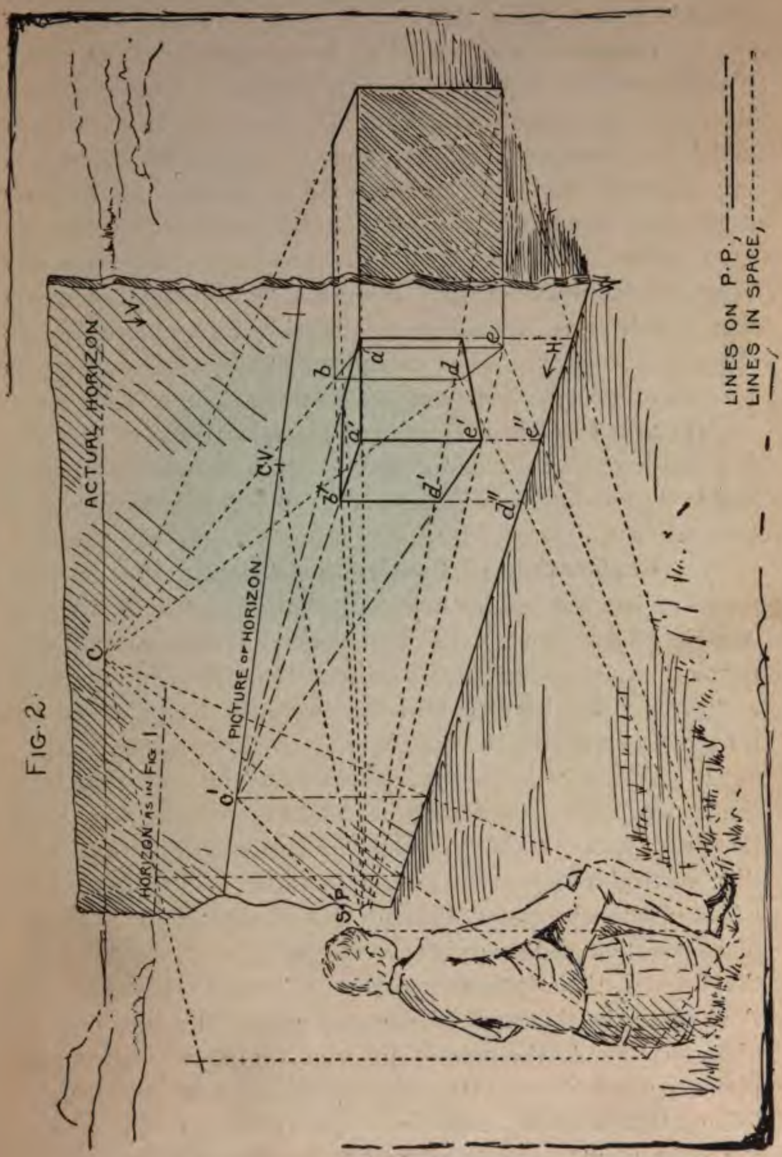

FIG. 2.

ACTUAL HORIZON
PICTURE OF HORIZON
HORIZON AS IN FIG. 1.
C
C.V.
O'
S.P.
V
H
a
b
d
e
a'
b'
d'
e'
d''
e''
LINES ON P.P.,
LINES IN SPACE,

The Picture-plane, or P.P., is the imaginary transparent plane in front of the artist.

The Center of Vision, or C.V., is that point on the picture-plane directly in front of where the artist is looking; it is at the foot of a perpendicular from S.P. to P.P. It is that point which is the center of a great circle of vision, formed by holding the head still and looking as far as possible in all directions, thus generating a cone of reflecting rays of light. It must not be confounded with the center of the picture as if the two were identical, for they are not. Only a small portion of that which is seen may be included within the limits of a sketch, and that small amount may lie principally above the horizon or principally below.

The Line of Sight is the perpendicular spoken of above, or line of direction S.P. C.V. It is of frequent use for reference and is thus distinguished from other lines of direction from the station-point.

12. Explanation of Construction-lines in Fig. 1.—The meaning has not yet been investigated of certain of the construction-lines shown in Fig. 1, namely, those dotted lines running from the feet of the artist to the points of the block, and those vertical dotted lines starting from the points of their intersection with the picture-plane.

In sketching this figure, it was found necessary to locate the exact points of intersection with the picture-plane of the imaginary radial lines from the observer's eye; in order to do this, vertical planes were conceived to be passed through the edges of the object and through the observer's eye; the lines of intersection of those planes with the ground are horizontal, the lines of their intersection with the picture-plane vertical. And since, in the case of any one plane, the line from the observer's eye to the point of the object must lie in the vertical plane passed through that point and the observer's eye, it follows that the radial lines before referred to can be cut off at the right points so as to represent their intersections with the picture-plane.

It may be seen also from this that each dotted line from the observer's eye to a point of the object lies directly over one of the dotted lines running from his feet to the object, and hence that e' and a' lie vertically over e', and likewise that d' and b' lie vertically over d'.

13. A Picture by Orthographic Projection.*—If an elevation is constructed of the picture-plane and points in it as shown in Fig. 1, looking in the direction of the arrow *H*, and a plan view in the direction of the arrow *V*, it is seen that the first gives the vertical heights of the various points in the picture, above the ground or horizontal plane, and the second gives the horizontal distances of the points, one from another, or from a convenient reference-point; in other words, a picture can be constructed of an object if the plan and elevation of it, together with the picture-plane and station-point, are given, by drawing the horizontal and vertical projection of lines from the points of the object to the station-point, in both plan and elevation, and noting their intersection with the picture-plane.

14. Problem.—It is now proposed to work a problem by this method, which may be described as a method of coordinates. It is not the usual way to make a linear perspective drawing, but it brings out clearly certain perspective features and moreover furnishes a very simple and satisfactory proof of the deductions previously made.

In Fig. 3, there is shown a plan and an elevation, respectively, of a truncated square pyramid, a picture-plane, and a station-point. The notations in the two views are distinguished in the usual way by the subscripts v and h respectively, meaning thereby vertical projection and horizontal projection.

If the line through the base of the pyramid in the vertical projection is regarded as a line of reference, say a ground-line,

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* If the student is not familiar with the elementary principles of orthographic projection, a very little investigation will render the process of the problem which follows clear, or it may be omitted altogether without interfering with consecutive treatment.

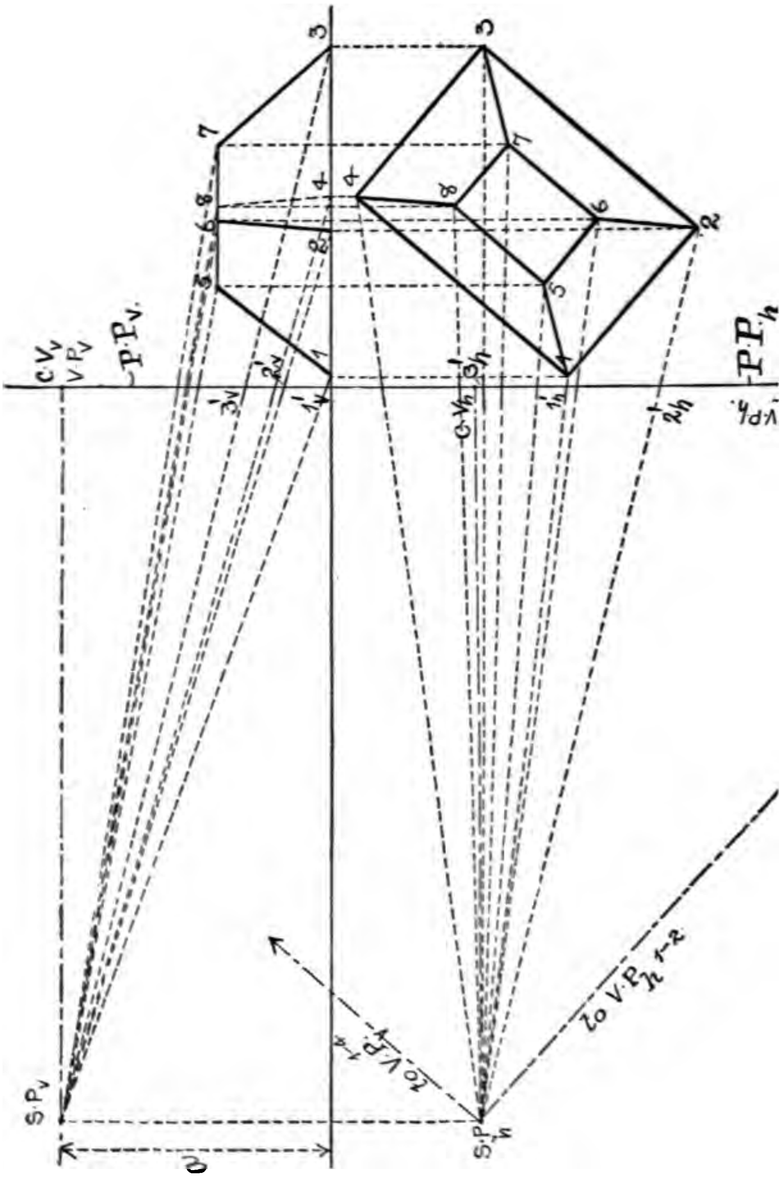

3
7
8
6
5
4
2
1
C·V_v
V·P_v
P·P_v
3'v
2'v
1'v
C·V_h
3'h
1'h
2'h
V·P_h
P·P_h
to V·P_h 1-2
to V·P_h 1-4
S·P_v
S·P_h
D

Fig.

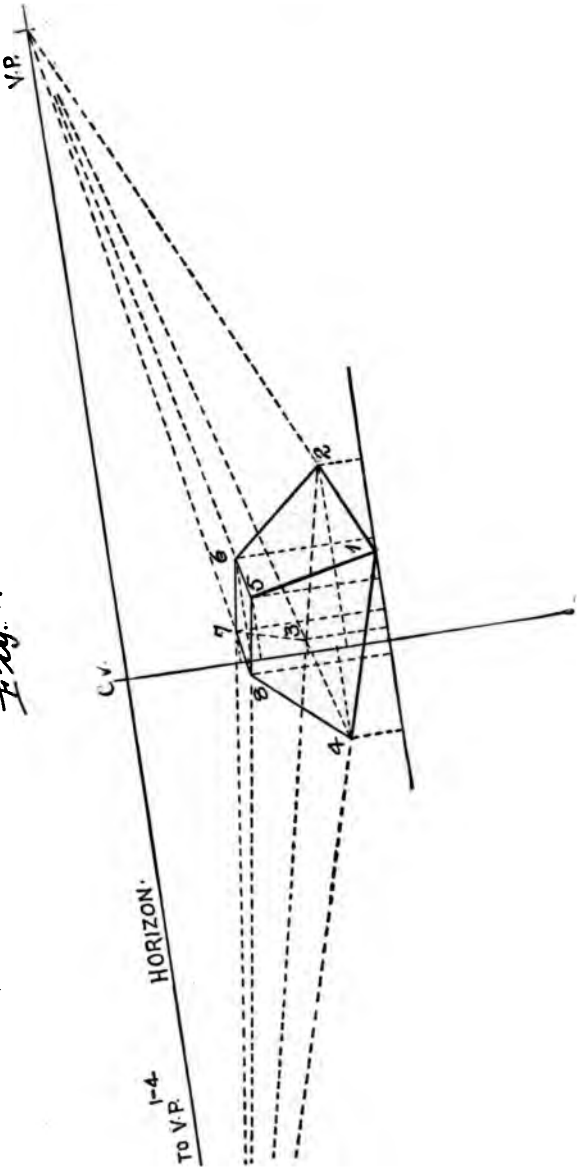


Fig. 4.

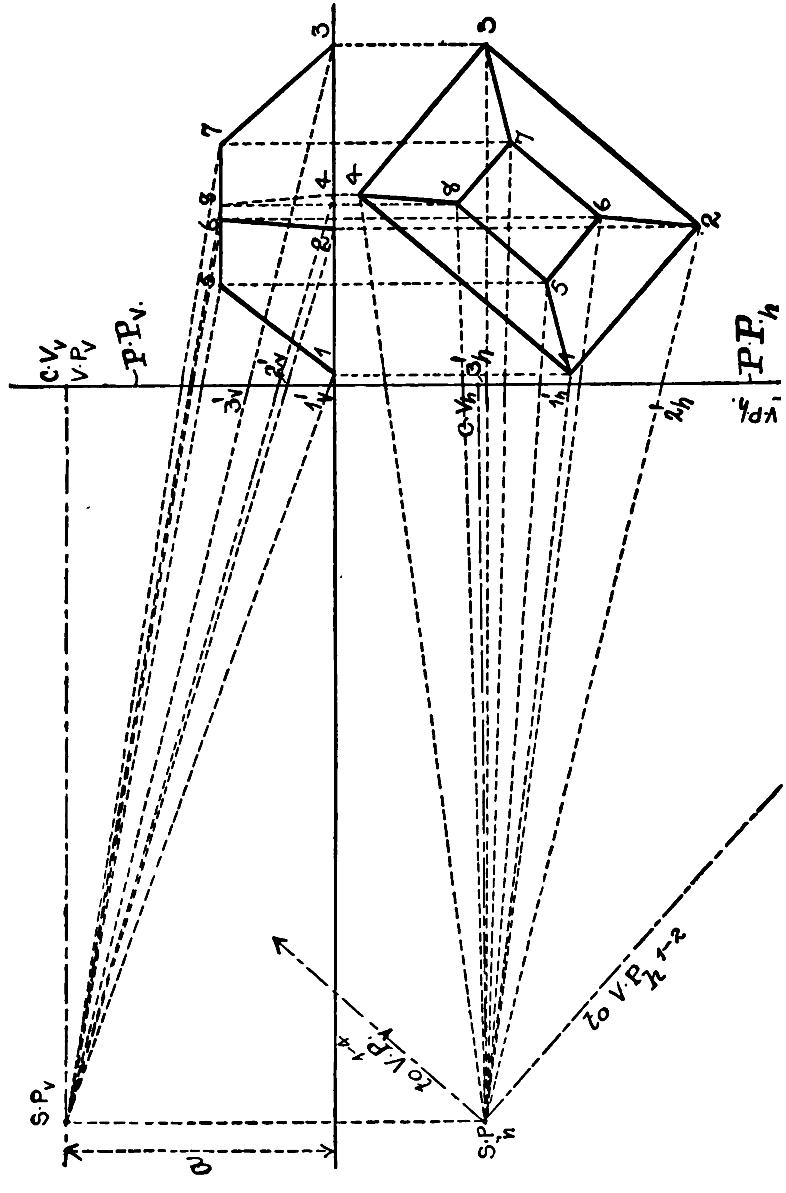
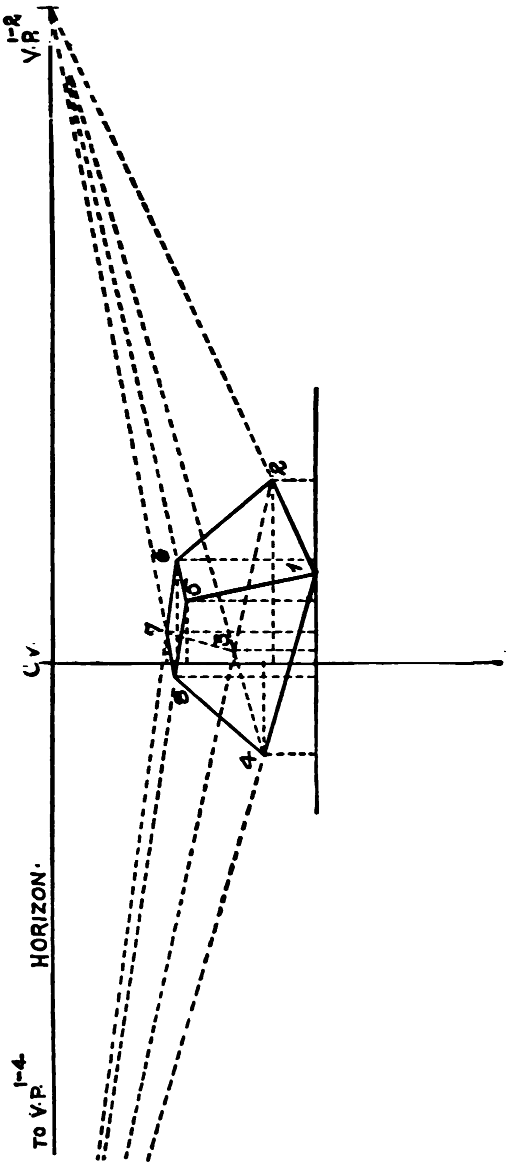

3
7
8
6
5
4
2
1
3
7
4
8
6
2
5
1
C·V_v
V·P_v
P·P_v
P·P_h
3'v
2'v
1'v
C·V_h
3'h
1'h
2'h
V·P_h
S·P_v
S·P_h
a
to V·P 1-4
to V·P_h 1-2

Fig. 4.



then the station-point is a distance *a* above the plane of the base, and if in the plan the horizontal projection of the perpendicular from the station-point to the picture-plane is regarded as a line of reference, then *S.P.* is a horizontal distance *b* from the corner of the pyramid nearest the picture-plane. Looking at the picture-plane from the observer's position, this would mean a distance *b* to the left of the nearest corner. Again, in the two views are shown the vertical and horizontal projections of the imaginary lines from the points of the object to the station-point. Hence it follows that the relative heights of the intersections of these various imaginary lines with the picture-plane are shown in elevation by the distances they cut the line of the picture-plane, respectively, from the horizontal line of reference, and likewise the relative distances of their intersections to the right or left of the observer are shown in the plan by the distances their intersections are from the horizontal line of reference $S.P._h$ $C.V._h$. In the two views, then, are included the positions of various points on the picture-plane referred to two lines as axes of reference; and these points are, in addition, the points of intersection of elements of a conical surface by the plane of the picture and of which the apex is at *S.P.* On a separate piece of paper the two axes can be constructed as in Fig. 4. The horizontal one represents the line of intersection of the picture-plane with the horizontal plane, and the vertical one represents the intersection on the picture-plane of a vertical plane passed through the station-point and perpendicular to the picture-plane. Its horizontal intersection would be indicated in Fig. 3 by the line *S.P. C.V.* Next lay off above the horizontal axis the distances obtained from the elevation which the various points 1′, 2′, 3′, etc., are from the plane of the base, then lay off the horizontal distances which these points are respectively from the vertical line of reference as obtained from the plan view, measuring to the right and left of the point *C.V.* The intersections of these two systems of lines give points in the outline picture of the pyramid as made on the picture-plane under the conditions assumed.

15. Proof of the Two Fundamental Laws of Perspective, *that parallel lines have a common vanishing-point, and the further an object recedes from the picture-plane the smaller it becomes.*—Now by the aid of these results a harmony can be proven with the laws previously deduced.

If the line $S.P._h$ $V.P._h$ in Fig. 3 is drawn parallel to the short edges of the base of the pyramid, it cuts the picture-plane a distance equal to $C.V._h$ $V.P._h$ to the right of the observer's position. If in Fig. 4 the edges parallel to 1 2 are prolonged, they will be found to intersect in the point *V.P.*, which is the same distance to the right of *C.V.*, and moreover, upon inspection, will be found to be as high above the horizontal plane upon which the pyramid is situated as the observer is. This distance can be verified from the elevation. The same construction might be applied to ascertain the truth as to the position of the vanishing-point of the edges 1 4 and 2 3, etc., but unfortunately the construction would run into the view above in Fig. 3, and perhaps be confusing. This is a proof by the method of orthographic projection of the rule for finding the vanishing-point of any line. The truth of the second deduction made, namely, that the further an object recedes from the observer the smaller it becomes, seems too self-evident from an inspection of the figures to need a demonstration in proof. Draw another picture-plane nearer the observer, or move the station-point further away, and ascertain what the effect would be.

CHAPTER II.

PLANES AND THEIR VANISHING-TRACES.

16. The Horizon.—If, while standing on the ocean beach on a calm day, one were to sight along a straight stick, pointing it out to sea at the far-distant horizon, it could be noted that it was being held practically level. The head might be depressed, or raised by standing on something, but the stick would still be level. The fact that the distance of the actual horizon, as compared with the measurements in an ordinary drawing, is very great, and also the fact that the surface of the earth, for the same reason, is practically level, makes it customary in perspective to at once use the statement that the horizon is the edge of an infinite horizontal plane.

17. The Common Trace of Parallel Planes.—Just as the vanishing-point of a line is the picture of that point of it which is at infinity, so the horizon in a drawing is the picture of that edge of a horizontal plane which is infinitely distant. And just as parallel lines have a common vanishing-point, so parallel planes will have a common vanishing-line. The line of intersection of two or more plane surfaces is generally spoken of as their trace; the horizon, in a picture, then, may be considered as the trace or line of intersection with the picture-plane of a plane passed through the station-point and parallel to the earth's surface. When, therefore, the vanishing-trace of any plane is spoken of, from now on, let it be understood to mean the picture of that edge of the plane which is infinitely distant, that is, its vanishing-edge.

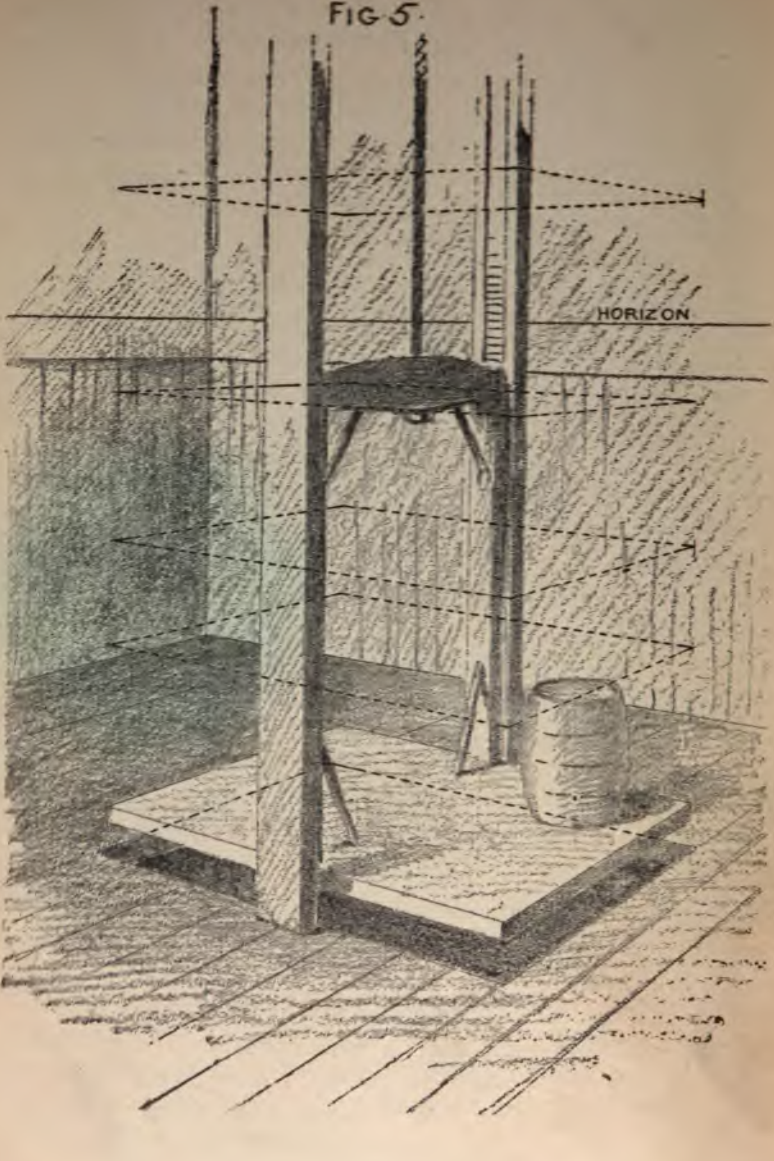
18. Illustrative Examples.—Standing within view of a moving freight-elevator, one in which the plane surface of the floor is visible, and watching it as it moves up through the floor, it can be seen that less and less of the surface is visible as it rises, until when it comes to the level of the eye, no floor at all is seen, only the edges show. In this position it would coincide with the line of the horizon if it were visible beyond. As the elevator continues to move up, the under surface comes into view, more and more of it showing if it is not interfered with by going through the floor above.

If a sketch were made of the plane of the elevator-floor in several of its successive positions, it would take the form of a series of rectangular figures one above the other, the receding edges of each of which converge more rapidly near the horizon, giving a smaller area to the enclosed figure; where the surface coincided with the horizon the plane would become a straight line. The vanishing-points of the receding edges remain fixed, but as the surface moves up from the floor they would make a smaller and smaller angle with the horizon, until they became coincident with it.

Fig. 5 is a drawing of such an elevator and shows by the dotted construction the general appearance of its floor in its different positions, and the apparent decreasing size of the surface and the changes in the angles which the receding edges make with the horizon.

Again, looking out of the window of a moving train, where the horizon is visible, perhaps a house, which is facing the track, seems to approach and its side wall can be seen quite fully and the front one, facing the track, very little. If the shape of the side wall could be drawn on the window-pane, it would appear a quadrilateral of considerable area. But as the train moves nearer to it the apparent area decreases. Just as the house is passed, it could be noted that, momentarily, the wall appears as a vertical edge, then disappears and the parallel wall opposite to it comes into view. Noting the position of the train when the first wall appears as an edge, it can be observed that the eye lies within the plane of the wall extended.

FIG. 5.



19. The C.V. Lies on the Vanishing-trace of Planes Perpendicular to the Picture-plane.—In section 11 attention was called to the fact that the point on the picture-plane directly in front of the observer was commonly known as the center of vision (C.V.). This point, it will be seen from the illustration cited, is a point common to the plane of the earth or horizontal plane, and the plane of the wall of the house, hence at the point of intersection of the vanishing-traces of the two planes. The picture of the vanishing-edge of the wall of the house is a line on the picture-plane perpendicular to the horizon and passing through C.V.

It makes no difference where any horizontal or vertical planes to be considered are, the picture of their vanishing-edges will be two right lines, one horizontal and one vertical. This can be very nicely illustrated in the floor, ceiling, and walls of an ordinary room. If the vertical planes considered are also perpendicular to the picture-plane, then the picture of their vanishing-edges will pass through C.V.

Obviously, if the horizon is the picture of the vanishing-edges of horizontal planes, it must contain the picture of the vanishing-points of all lines lying in such planes, namely, all horizontal lines.

20. General Relation of Lines and Planes.—In geometry, lines are invariably associated with planes; so, in perspective, it is of great assistance, to the proper understanding of many of the practical problems that occur, to consider lines in the same way, either by the planes within view, or, for convenience, those which may be passed through them. What is true of the relation of the horizon to horizontal lines is also true of any plane no matter what its direction; so to make the statement perfectly general: *the trace or line of intersection with the P.P. of a plane passed through the observer's eye will contain the picture of the vanishing-points of all lines lying in planes actually parallel to that imaginary one.* It follows readily from this and from what was said regarding the traces of horizontal and vertical planes in section 19, that if two plane surfaces intersect, the picture of the vanishing-point of their line of intersection, which line is common to the two planes, will be that point in which the traces of two planes passed through the S.P. and parallel respectively to the two planes considered intersect each other on the P.P.* In Fig. 6, for example, the edge *AB* is common to the two plane surfaces *ABE* and *ABCD*, and its vanishing-point is at the intersection on the P.P. of the traces of the two planes, namely, *V.P.*

21. Some Familiar Planes.—Horizontal and vertical planes are familiar to every one because the things about us, buildings and other works of man, owe their stability to and consist chiefly of such. It is for this reason that attention is first called to them in preference to others. They are the planes which in pictures are apt to be most readily separated from others and criticised. One would find fault with a picture the instant the walls of a house did not look plumb or a floor level. Furthermore, any one can very easily detect the slightest variation in direction from the vertical or horizontal no matter from what point of view he is looking at the lines considered.

22. To Ascertain the Traces of Planes in any Picture.—Suppose now it is desired to ascertain where the horizon is in any given picture, where there may be buildings shown. Knowing that it must be horizontal, all that is necessary to do is to prolong any two parallel horizontal lines, such as the base-line and eaves-line of a house or the lines of the window-sills, until they meet in a point; this point will be in the horizon. If it be desired to find the picture of the vanishing-edge of any other plane visible, which is oblique to the horizon, such as a sloping roof, then, remembering from geometry that two points will fix a straight line, prolong any two different sets or systems of parallel lines which are known to lie either in the plane of the roof or in one parallel to it; these two sets would meet in two points which, being in the plane, would be sufficient to locate its trace. It is not possible to employ the same rule for finding the vanishing-trace of a plane that is used for finding the vanishing-point of a line, as given in section 7, because a plane has extent in all directions within its surface; but it is possible to find the vanishing-points of two or more lines within it, and these will decide its trace.

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* It does not follow, however, as in the case of the traces of horizontal and vertical planes, that the angle between the vanishing-traces of any two planes is a measure of the angle that the planes are to each other or to the picture-plane.

23. **Illustrative Problem.**—In Fig. 6 is shown a sketch of a block, and let it be assumed, to begin with, that nothing is present on the paper but the block correctly drawn. It is desired to locate the picture of the vanishing-edges of the several plane surfaces composing it. It has one horizontal rectangular face *ABCD*, three vertical faces, and one inclined face *AEFD*; prolong either of the two sets of parallels, *AB* and *DC*, or *BC* and *AD*, to meet in either $V.P.^1$ or $V.P.^2$, and by means of either of these the vanishing-trace of the horizontal base is located. Since the face *AEB* is vertical, and since $V.P.^1$ is one point in its trace, that trace must be a line perpendicular to the horizon through $V.P.^1$ And for the same reason the trace of the face *BEFC* must be a vertical line through $V.P.^2$ Now the inclined face has two edges which vanish in $V.P.^2$, so that the trace of its plane must pass through this point, and it, moreover, has one edge lying in the plane *ABE* and one edge *DF* parallel to it in the plane *DCF*; hence, by prolonging these two edges, they will meet in a point in the vertical trace through $V.P.^1$, namely, $V.P.^3$. Therefore the trace of the inclined face is located by drawing a line through $V.P.^2$ and $V.P.^3$

24. **Changes to be Observed when the Object is Rotated or Moved.**—In Fig. 7 the same block is shown as standing on one of its edges, and the face *AEFD*, which before was inclined, is now horizontal. The block has been rotated, it may be assumed, on the rear edge *BC*. In this rotation the edges *BC*, *AD*, and *EF* are still horizontal, and parallel to their former positions, and will continue to vanish in $V.P.^2$

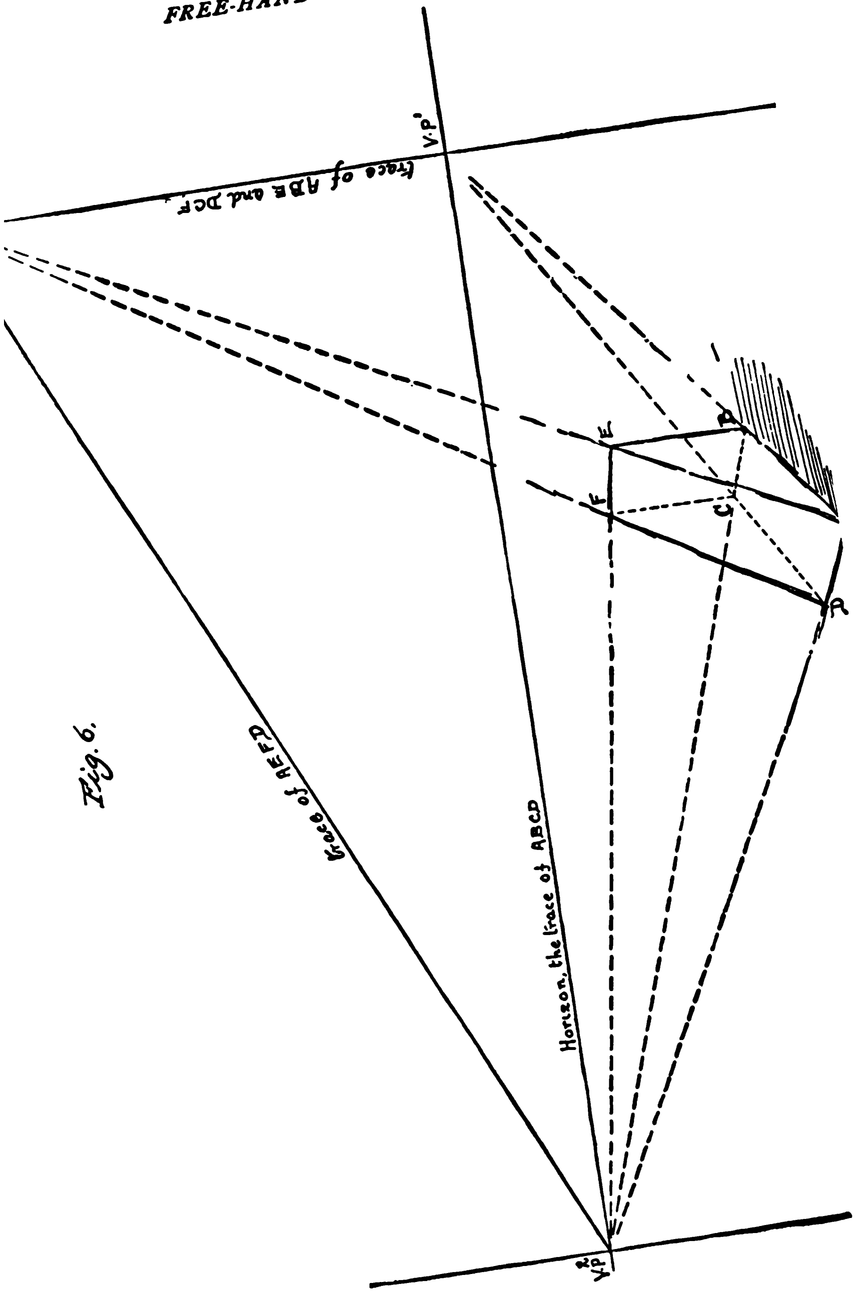

V.P¹
trace of ABE and DCF
E
D
F
C
A
Trace of AEFD
Horizon, the trace of ABCD
V.P²

Fig. 6.

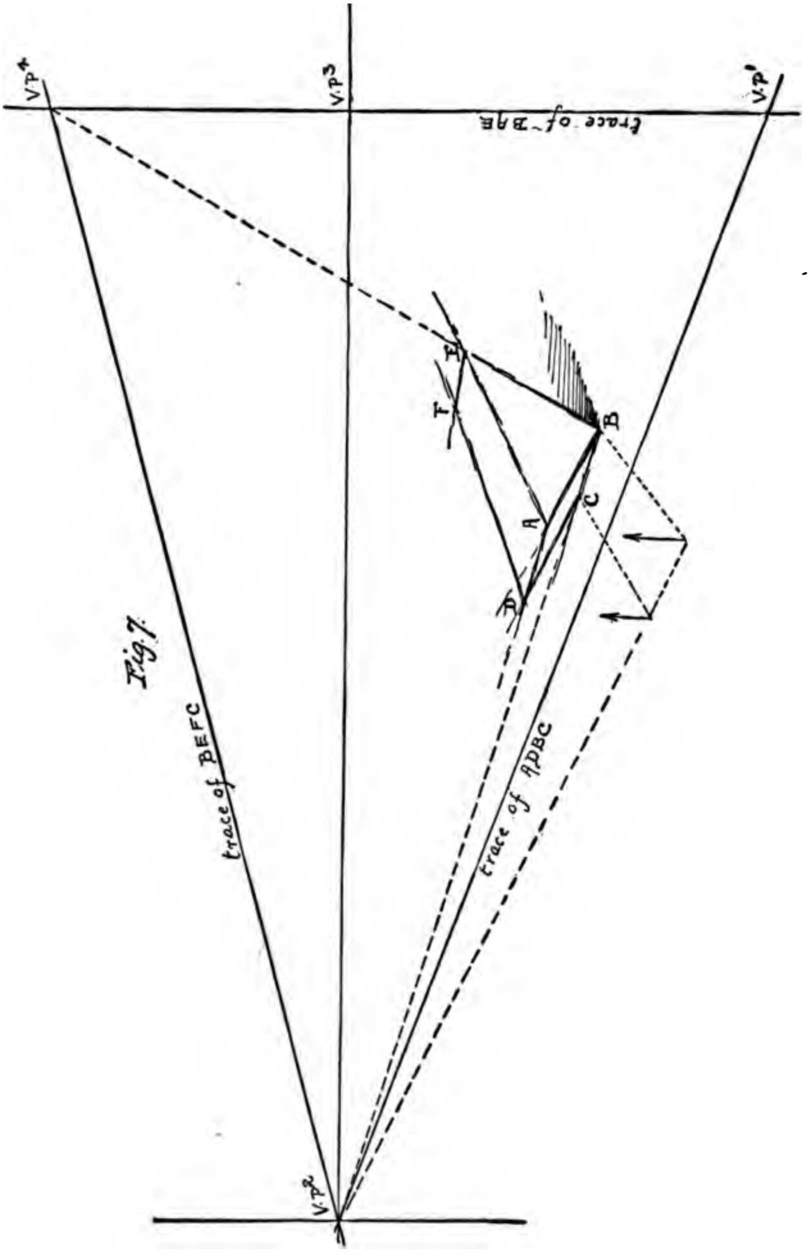


Fig. 7.

V.P⁴
V.P³
V.P¹
trace of ABE
E
F
B
C
A
D
trace of BEFC
trace of ADBC
V.P²

The vertical faces *BAE* and *CDF* are still vertical, hence their trace will be the same line as before, this time cutting the horizon in *V.P.*[3], which latter point is found by prolonging *AE* and *DF*. *V.P.*[3], in the rotation, has moved down to the position occupied by *V.P.*[1] in the first figure, since the plane *AEFD* is now horizontal. Similarly, *V.P.*[1] has moved to the position shown. The traces of the planes *BCAD* and *BEFC* can now be located, because it is known that *V.P.*[2] is one point in both traces, and that the other points must be somewhere in the vertical line through *V.P.*[3] and found by prolonging the two parallel edges in each plane.

If the block from its position in Fig. 7 were moved upward so that its surfaces remained parallel in their successive positions, then, when it reached the horizon, the upper surface would be seen as a straight line. If the block were moved still further, this surface disappears and, obviously, *BADC* would appear larger. If, instead of this, the block be moved downward in the same manner, this face *BADC* would grow apparently smaller, and it in turn would be seen as a straight line when it reached the position of the line *V.P.*[2] *V.P.*[1] It is here left to the student to answer the question what simple motion and to where would cause the face *BEA* to appear as a straight line.

25. Illustrative Problems.—In order to fix in mind the facts herein enumerated with regard to lines and planes, it would be well to work one or two problems by the method described in section 22, and for this purpose several are subjoined.

It may be well to state, however, before beginning them, that difficulties may be encountered, in such problems as these, in realizing the true meeting-point of parallel lines. It is impossible to attain absolutely exact construction, but approximation to the truth may be close enough to satisfy all reasonable demands; when it is realized that all sketch-work is approximate, it will easily be seen that to adjust the sketch to fulfil the evident conditions is a legitimate part of the problem ;

therefore, in those that are to follow, adjustments should be made in conformity with the known facts of each.

Problem 1. In the middle of the space on a sheet of paper 13″ × 13″ either copy the drawing of the truncated prism Fig. 8*a* carefully or else make a tracing and transfer it by a sharp point and transfer-paper. Then ascertain the traces of all the planes composing the subject. Letter the faces of the figure *A*, *B*, *C*, *D*, etc., and when found letter the traces of the planes to correspond, *T.A.*, *T.B.*, *T.C.*, etc.

FIG 8a

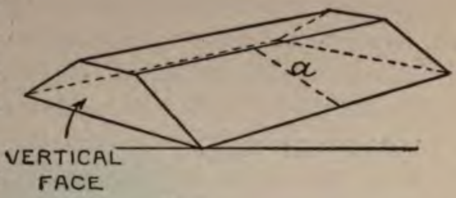
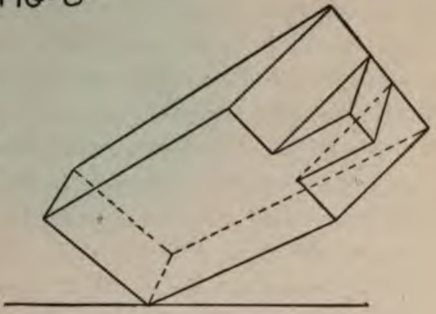
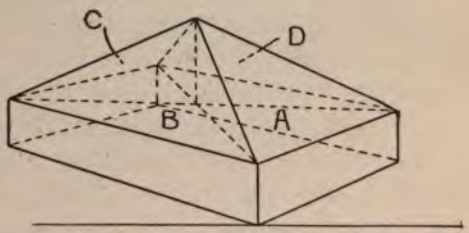


FIG. 8b



In addition to the above sketch a horizontal plane through the middle, removing the upper half; also pass any plane through the dotted line *a*. If the first part of the problem has been done lightly in pencil, then the necessary changes can be made without redrawing.

FIG 8c



Problem 2. In a similar manner copy the tongue-joint of

Fig. 8*b* in the center of a space 14″ × 14″ and find the traces of all the planes composing the subject, and letter as in problem 1.

Problem 3. Copy the pyramidal block of Fig. 8*c* in the middle of a space 12″ × 12″. Find the vanishing-traces of all the planes, and letter as in problem 1.

In connection with the last problem, it will be instructive to find the answer to the question why do the traces of the planes *B* and *D* intersect on the horizon? A hint to the answer may be found in section 20. Also, what horizontal motion of the block, and to where, would cause the plane *D* to be seen as a straight line ?

26. Generated Lines and Planes.—As an aid to the understanding of lines and planes and their various relations in perspective, it will be well to adopt a mathematical conception, namely, that lines may be generated by moving points, and planes by moving lines.

If a geometrical point moves along a path, then this path will be a geometrical line, varying in its character according to the character of the motion. It is also known as the locus of the various positions of the point. If the direction of the motion is constant, then the path or locus is a straight line. If the direction is continually changing, then the locus is a curve. If motion is always in one plane, then the path generated is a plane curve.

If a geometric line is conceived to move, it will generate a geometric surface, which may be plane or curved according to the law of the motion. The line may move so that its successive positions are parallels, or it may rotate about a point or points in space or within the line itself. If it move in a fixed direction either parallel to itself or rotate in a plane about a point within it, then it will generate a plane surface. If the line is indefinite in length, then the surface will extend indefinitely; but if it is of finite length, then the limiting edge of the generated plane surface contains the loci of the moving extremities or points of the line. By having a changing direction, the generating line will describe curved surfaces of various kinds, according to the law of its motion.

This reasoning can be found exemplified in Figs. 6 and 7. In rotating the block on the axis *BC*, the line *AE* generated a vertical plane in the plane of *ABE*, similarly *DF* in the plane of *DCF*, and the picture of the vanishing-point of *AE* and *DF* moves in the straight line $V.P.^3$ $V.P.^1$ In other words, if the observer, from his position, looks in the direction of the edge *AE* in its position in Fig. 6 and then moves his line of direction, as *AE* moves in the rotation, following the motion of, and always parallel to it, this line will intersect the P.P. in successive points which constitute the trace $V.P.^3$ $V.P.^1$

27. Examples of Generated Surfaces.—A few examples of the various surfaces which may be generated by a moving line are given below:

A cone, which would be termed a cone of revolution, is formed by a line which rotates about one of its points, and not in one plane or in a constantly fixed direction. The kinds of conical surfaces generated will depend upon the relation of the generating line to a second line which is known as the axis of the cone.

A circular cylinder of revolution is one in which the generating line moves parallel to itself and at a fixed distance from a second line or axis. Various cylinders can be generated according to the relation of the generating line to the axis.

28. Axis of a Plane.—Furthermore, it is useful sometimes to refer to the axis of a plane. The axis of a plane is any line which is perpendicular to the plane, if the latter is indefinite in extent. If it is a limited plane, then the axis must not only be perpendicular, but must go through a particular point known as the center of gravity of the plane. In Fig. 6, for example, *BC* may be taken as the axis of the plane containing the triangle *ABE* and which has its trace in $V.P.^3$ $V.P.^1$ The axis of the triangle would have its vanishing-point in the same place as any axis of the indefinite plane containing the triangle. The axis of the plane *BEFC* would go through the center of that rectangular figure and vanish in *V.P.*[1] The use of the term axis will have more significance later.

29. A Line has Generally One Vanishing-point and a Plane One Vanishing-trace.—From what has gone before, it will be readily seen that a line will have but one vanishing-point when it is inclined at a greater angle than zero degrees to the P.P., and that point is in the direction of that end of the line which is furthest from the observer. When the line is parallel to the P.P. it has two vanishing-points, but at infinity in the direction of either extremity. A plane has but one trace if it is inclined to the P.P., and that is a straight line. In case the plane is parallel to the P.P., its trace, which is at infinity, is no longer considered, although it may be thought of as a great circle.

30. A Few Geometrical Facts with Regard to Lines and Planes.—Before leaving the subject of planes it would be well to bear in mind the few following facts of geometry with regard to lines and planes, as they will be quite useful in future problems to be taken up:

(*a*) An infinite number of planes can be passed through a line, but only one having a given direction.

(*b*) Only one plane can be passed through two lines, and to do this, moreover, the two lines must either be parallel or intersect.

(*c*) An infinite number of planes can be passed perpendicular to a line, and but one through a given point in it.

(*d*) An infinite number of planes can be passed parallel to a line. If all the planes are at the same fixed distance from the line, they will envelop a circular cylindrical surface.

Note.—The planes in (*a*) will be perpendicular to those in (*c*).

CHAPTER III.

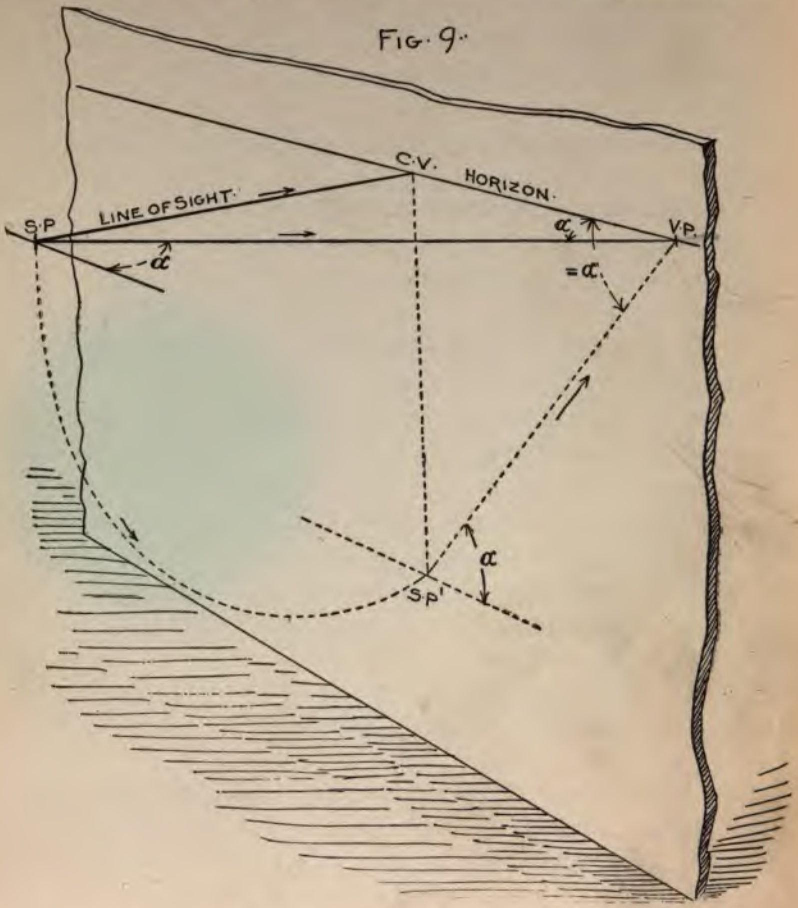
TO FIND THE VANISHING-POINTS OF LINES.

31. Lines as Associated with Planes.—We now come to a consideration of those processes which make it possible to get definitely the positions and lengths of lines in space. It is not here intended to make the treatment complete; only that consideration will be given it which seems necessary for use in sketch methods.

The first thing to be done in considering a line and its vanishing-point in perspective is to associate it with some plane or planes. This may be very conveniently settled by the fact that it lies in the same plane with other lines whose vanishing-points are known, or it may be the line of intersection of two or more planes whose traces are known, or, on the other hand, it might be an isolated line through which an infinite number of planes may be passed. Since horizonta lines and planes present perhaps the least perspective difficulties, owing to our familiarity with them, we will confine our attention to them for the present.

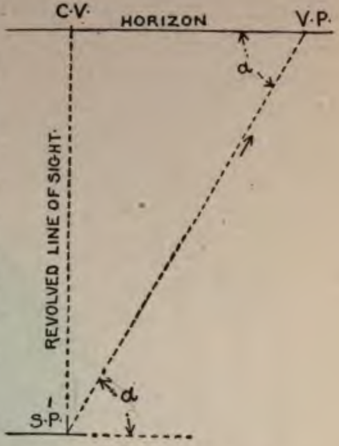
32. Rotation of the S.P. into the Plane of the Picture.—The rule for finding the vanishing-point of a line has been given, but it remains to apply that rule to any drawing. It is not convenient to run imaginary lines in space to intersect an imaginary plane, such as is pictured in Fig. 1. The constructive work must be so arranged that it can be recorded entirely upon a sheet of paper; to do this the following device is used which will, for the present, be applied to horizontal lines: If a point is rotated in a plane perpendicular to and about any line, then its relations to that line in its various positions will be the same. In Fig. 9 is a picture similar to Fig. 1; the line *S.P. V.P.* is a line of direction assumed to be actually parallel to some line at which the observer is looking through the

FIG. 9.



picture-plane. If now the *S.P.* is rotated about the line of the horizon into the plane of the picture and in a plane perpendicular to the horizon, namely, one which passes through it and *C.V.*, the line of direction just spoken of will have the position *S.P.*[1] *V.P.* It will make the same angle with the horizon in this position that it did when it stood in the plane *V.P. S.P. C.V.*, which is perpendicular to the picture-plane, that is, the angle α is the same in both cases. Moreover, from geometry: *the alternate interior angles of parallel lines are equal;* hence the angle which this line of direction makes with a line through the *S.P.* and parallel to the *P.P.* (and the horizon) will be equal to the angle the line makes with the *P.P.* as shown, and likewise will be equal to the corresponding angle at $S.P.^1$ The construction as it would ordinarily be made on paper is that shown in Fig. 10.

.Fig. 10

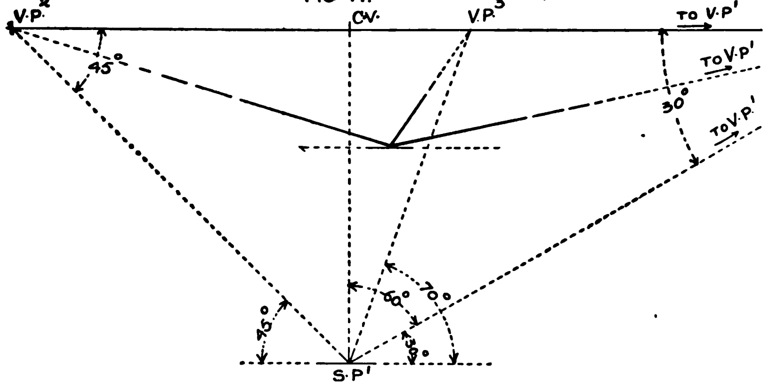


This is the process which is used in finding the vanishing-points of any horizontal lines. For example, if the angle that any certain horizontal line makes with the picture-plane is known, then a line of direction can be drawn from the rotated position of the station-point, $S.P.^1$, making the required angle with a parallel to the horizon, and where this line of direction cuts the horizon will be the picture of the vanishing-point. It will be that point in which the same line of direction would cut the picture-plane if it were drawn from the observer's position, *S.P.* (see Fig. 9).

It is to be noted that the radius of the arc of rotation of the *S.P.* in this case is a line from the *S.P.* to *C.V.* which is both perpendicular to the horizon and to the *P.P.*; in other words, it is the distance which the observer is from the *P.P.* The direction of rotation which is taken is that usual in perspective, though there is nothing to prevent its being rotated into a position above the horizon.

33. Illustrative Example.—Fig. 11 shows the practical and diagrammatic construction for ascertaining the vanishing-points of several different horizontal lines making several different angles with the horizon and the picture-plane. One line makes an angle of 30°, and the line of direction to find its vanishing-point is drawn at that angle to the horizon. *V.P.* will be its vanishing-point. Another line makes an angle of 70° with the *P.P.*, and its vanishing-point is similarly found, and so also with the one making an angle of 45°. Now radially from a point which is assumed in the neighborhood of the *C.V.* may be seen several limited lines, that is, of any finite length, which are the pictures of lines having the angles with the *P.P.* as shown. It may be seen in the case of any line, for example the one which is at 30° to the *P.P.*, that its position might have been designated instead by the angle it made with the perpendicular from the *S.P.*[1] to the *P.P.*, a line which has before been described as '*the line of sight.*'

FIG. 11.



34. The Distance-point.—It is also to be noted that in the case of the line at 45° to the *P.P.*, *V.P.*[2] is a distance from *C.V.* equal to the distance which *S.P.*[1] is from *C.V.*, namely, this vanishing-point shows the distance the observer is from the *P.P.* For this reason it is often spoken of as the '*distance-point.*'

35. The Application of Perspective Principles to Sketching.—Before taking up the further application of these principles to problems, it will be well to consider briefly the relation of perspective to the general uses of the artist. The application of perspective to sketching is accomplished by a succession of approximations just as a drawing is made, for example, of a group of geometrical solids, in a classroom, by the students. The most approved method is, not to draw line for line with an exhaustive care as to the direction and character of each separate line used, but the general impression of the whole is recorded in as few suggestive lines as possible, and the suggestion developed and refined by a succession of approximations closer and closer to the impression which the subject makes upon the observer. If, in conjunction with this method of sketching, the student or draftsman can apply a perspective fact here and there to help in the record of impressions, it will be a perfectly legitimate and added assistance.

If it is desired to make a drawing in outline, let us say, of something which is not available to follow, then the only thing possible is to depend upon knowledge of the appearance of things gained from an experience in sketching together with a command of perspective principles. A rough sketch is made, somewhat at random, and to this approximation perspective tests are applied in order to correct the work, so that the drawing has the effect which it would have were it drawn on a transparent plane between the artist and the object. In general the artist would be very little concerned with the exact positions of the vanishing-points of lines, though it is necessary to understand where they would be were he correct in his estimate. Likewise with the measurements of the sizes of parts, he cares very little about the geometrical exactness of his estimate; the true impression upon any observer is what is striven for. This is the spirit in which the subject will hereafter be treated in this book as far as it is possible to apply it to the study of principles which are more or less rigid, and in all cases founded upon geometrical facts.

36. Problem of Sketching a Square.—It is rather difficult for a beginner to realize the conditions in constructive work involving merely lines not associated with any familiar form or object; therefore in the few problems dealing with the vanishing-points of lines a square plane will be used as a basis. Now it is possible even without a knowledge of perspective, yet with a little practice in drawing, to make a fairly good sketch of a square; perspectively it is not possible to go very far wrong in it at any rate, for with slight exceptions any four lines one chooses to draw enclosing a polygon will be the possible picture of a square seen from some point of view. The perspective conditions may not be easily realized, or, if realized, good ones to use for a picture of the square, but they will be true nevertheless; the reason for this can easily be seen: the edges of the square can have but two vanishing-points, and two points are sufficient to locate the vanishing-trace of the plane of the square, and it is possible by shifting the observer's position to make the trace of a plane through S.P. and parallel to the square intersect the picture-plane in the right place.

37. Problem 1. A Horizontal Square.—Let it be required to sketch a horizontal square, and let the conditions be that the edges of the square make equal angles with P.P. and that the plane of the square is below the eye; the other conditions and measurements are left optional. This is one of the simplest cases (see Fig. 12).

Assume a horizontal line anywhere as the horizon or vanishing-trace of the plane of the square, and assume $S.P.^1$ any convenient distance below it (remember that this is the rotated position of the $S.P.$). If lines of direction are drawn from $S.P.^1$ making angles of 45° with a parallel to the horizon, then their intersections with the horizon in $V.P.^1$ and $V.P.^2$ will be the vanishing-points of the edges of the square. Next assume A, the nearest corner of the square, any convenient place, preferably relatively near $C.V.$ (which it will be remembered is at the foot of a perpendicular from $S.P.$ and $S.P.^1$ to the horizon), and sketch lines to $V.P.^1$ and $V.P.^2$ respectively. AB, a side of the square, can be taken any convenient length. Now since the edges of the square make equal angles with the

P.P., one diagonal of the square must be parallel to it, and from this the adjacent side of the square can be cut off at *D*; from *B* and *D* lines drawn to the two vanishing-points will complete the square. As a test of the correctness of the sketch it can be observed that the other diagonal of the square will be perpendicular to the *P.P.* and must vanish in *C.V.*

Had the edges of the square made unequal angles with the *P.P.*, then the measurements of their lengths would not have been such an easy matter (the question of how to measure them will be taken up later). The use of a little judgment in the application of geometry to the problem makes it possible to measure the sides without special perspective methods; and it is well to state here that short cuts are in place if they are based upon geometrical principles.

38. Another Method.—Before leaving this problem it would be well to look at another way to proceed with it; that is, instead of assuming a horizon and vanishing-points which at once limit the perspective conditions of the square, suppose a sketch is first roughly made by an approximate guess at the form required. The first method would do very well if the thing to be drawn were as easily done as a square, but were it an intricate subject the result of empirically fixing the conditions of the observer's position relative to it would often lead to results far from satisfactory, though they may be true perspectively. Start at the corner *A* and sketch indefinite lines going through *B* and *D*; in so doing be careful to make the inclination upward of each somewhat nearly equal. Exactness in this respect is not at all necessary, as will be seen later. The angle of inclination of these lines can also vary according as it is desired to have the square near the level of the eye or below it; their angle with the horizon will increase as the square is further and further below the eye; it is best, however, to make it quite small. Next assume *B* anywhere which will give the sized square desired, and draw the horizontal diagonal *DB*. The remaining two sides of the square are parallel to the first and can be sketched to appear to converge to vanishing-

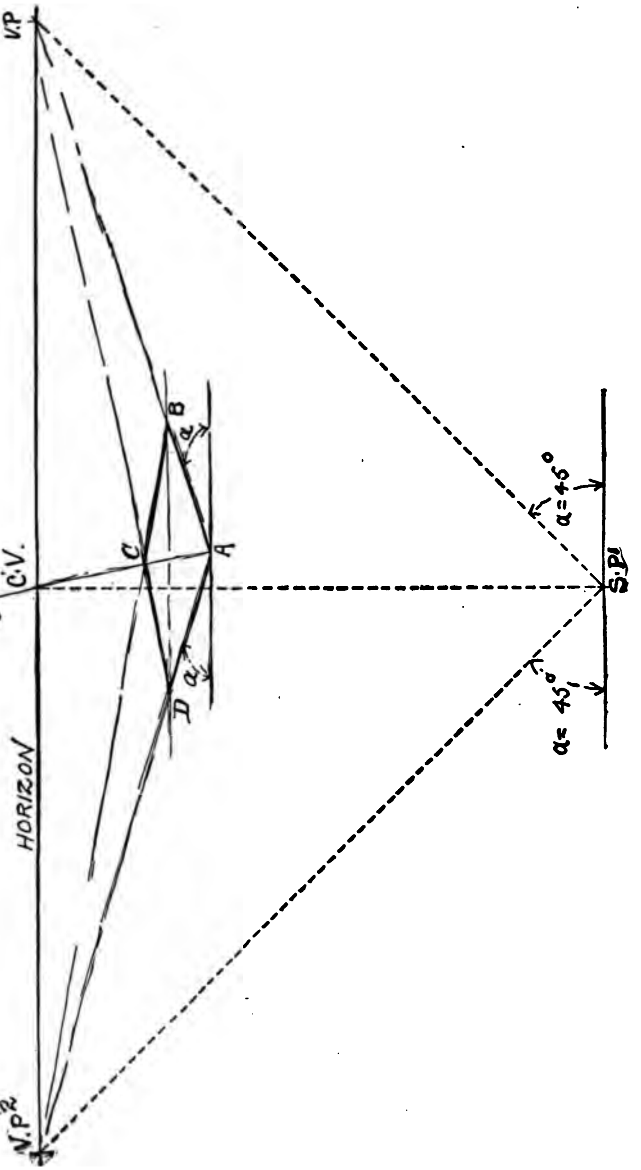


Fig. 12

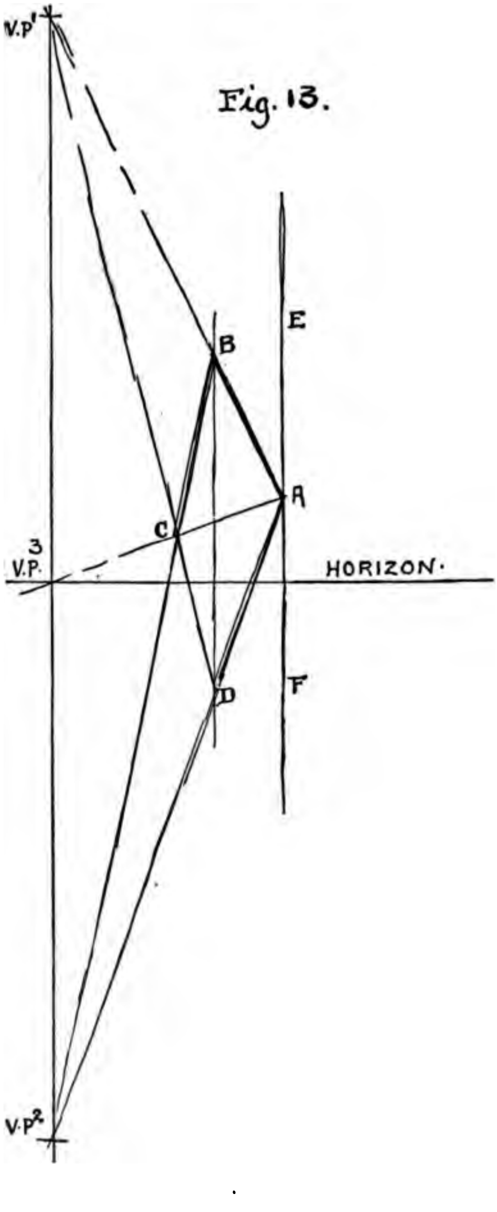
points. To ascertain whether the sketch is sufficiently correct to be acceptable, the two vanishing-points last found must be so situated that a horizontal line can be drawn through them; if they do not stand this test, then a horizon can be sketched through either one, and the other dropped or raised to meet it, the corresponding lines of the square being changed, or the horizon can be taken between the two, whichever seems to give the most pleasing result. Next the *C. V.* will lie midway of these two vanishing-points, and the second diagonal should go through this point. If a perpendicular to the horizon is erected at *C. V.*, the rotated position of the *S. P.* must lie on it a distance from the horizon equal to the distance of *C. V.* from either vanishing-point. Whether this perpendicular passes through *A* or slightly to the right or left, merely indicates whether the square was directly in front of the observer with respect to the *P.P.* or slightly to the right or left, and if it is not too far either way, will be satisfactory enough to leave as it is. Had the angles originally been drawn exactly equal, then the perpendicular would have passed through *A*. Again, if the two vanishing-points had happened to come further apart it would simply have meant that the observer was assumed further away from the *P.P.* when sketching.

This last method would be the artist's way in most cases of applying his perspective, for he can as a rule trust his mental picture of what he wishes the result to look like better than he can locate vanishing-points with a clear conception of what his sketch will result in by a strict conformity to them.

39. Problem 2. A Vertical Square.—Let it be required to sketch a square plane, the only conditions imposed being that it shall stand in a vertical plane with edges equally inclined to the horizontal plane.

If so inclined, they must also be equally inclined to a vertical plane, hence to a vertical line; this is shown in Fig. 13. The vertical line *EF* may be assumed to be the line in which the plane of the square intersects the *P.P.* The vanishing-trace of the plane of the square will also be a vertical line

Fig. 13.

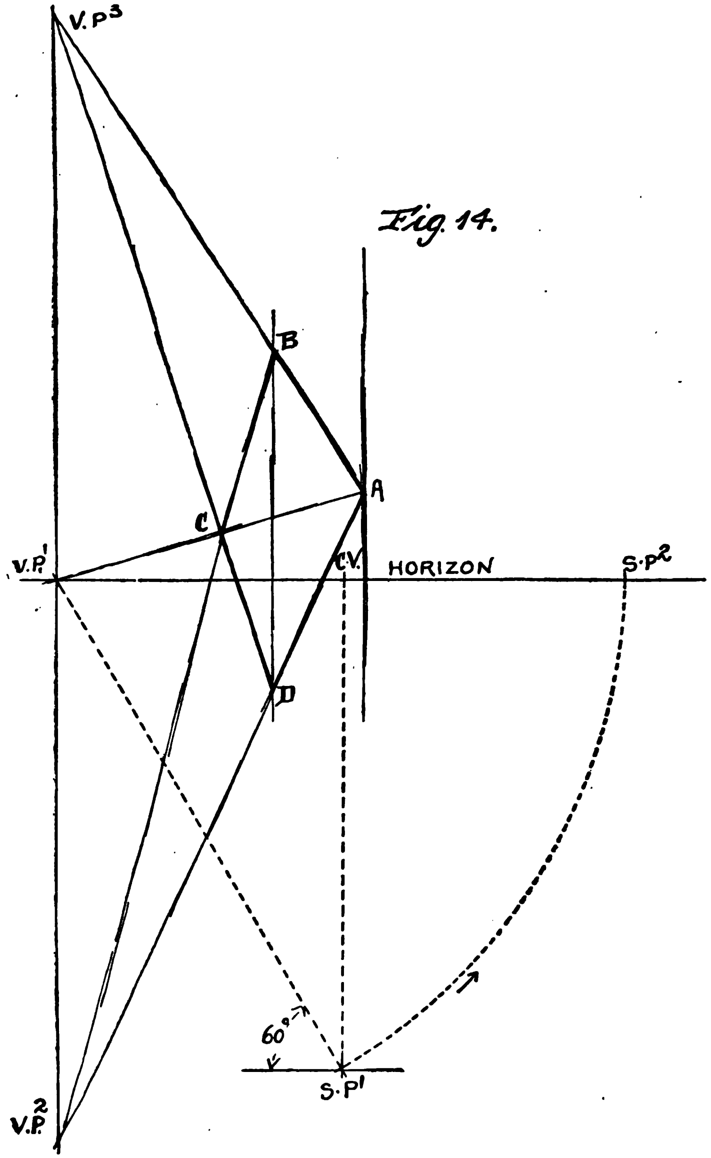


and parallel to this one. This can first be sketched anywhere, and a horizon to cross it. The receding edges of the square will have vanishing-points in the vanishing-trace of the plane of the square equally distant above and below the horizon. The nearest corner of the square *A* can next be assumed (preferably near the junction of the horizon and the trace), and the remainder of the drawing completed as in the previous problem. i.e., one diagonal, *DB*, which would be first drawn, is vertical, and the other, *AC*, is horizontal and must vanish in *V.P.*[3], lying as it does in the two planes, the horizontal plane and the plane of the square. In this case it is not necessary to know the position of *S.P.*[1], not being concerned with the angle which the plane of the square makes with the *P.P.*

It would be of course possible to proceed similarly to the method described in section 38, that is, to sketch the square first by an approximate guess. Here the angles *EAB* and *FAD* should be made approximately equal.

40. Problem 3. A Vertical Square at a Definite Angle to the P.P.—Sketch a square vertical plane, its edges equally inclined to the horizontal plane and the plane of the square at 60° to the P.P.

To do this problem by the first of the two methods described in problem 1, it is necessary to assume first the horizon and a station-point (see Fig. 14). Since the plane of the square is vertical, a horizontal line in it can be used to measure its angle with the *P.P.*; in other words, it is necessary to construct a horizontal line of direction from the observer's position parallel to the plane of the square. This by section 33 can be done at *S.P.*[1], giving *V.P.*[1]; the vanishing-trace of the plane of the square will be a vertical line through this point. The edges of the square will recede to vanishing-points equally distant above and below the horizon; but since *S.P.*[1] has been first assumed, they cannot as before be taken at random: the *S.P.* will govern their location. In this case the line of direction *S.P.*[1] *V.P.*[1] must be rotated about the trace of the plane of the square until it falls on the *P.P.*, namely, on the horizon at


V.P³
Fig. 14.
B
A
C
V.P¹
C.V.
HORIZON
S.P²
D
60°
S.P¹
V.P²

S.P.[2] The student must bear in mind always that *S.P.*[1] is the rotated position of the observer, and that his true position is in front of the *P.P.*, and in the rotation last described the motion is in a horizontal plane about *V.P.*[1] After this lines of direction can be drawn from *S.P.*[2] inclined at 45° each to the horizon, giving *V.P.*[2] and *V.P.*[3] as the vanishing-points of the receding edges of the square.

The second method, namely, sketching offhand the square first to fulfil the conditions by eye, cannot be taken up now because it involves an estimation of the values of plane angles, which has not yet been touched upon.

41. Problem 4. A Square Plane Perpendicular to the P.P., but Inclined to the Horizontal Plane.—Sketch a square whose edges are equally inclined to, and whose plane is perpendicular to, the P.P., but making a definite angle, 30°, with the horizontal plane.

As the plane of the square is perpendicular to the *P.P.*, its trace must pass through *C.V.* (see Fig. 15);* moreover, the angle which its trace makes with the horizon will be the true angle which the plane of the square makes with the horizontal plane, for the line of intersection of the plane of the square and the horizontal plane is parallel to the line of direction *S.P.*[1] *C.V.* Hence in this problem it is not necessary to locate the observer's position, but nevertheless it will be assumed as shown, so as to lead to an understanding of the rotation which would be required to get from it *V.P.*[1] and *V.P.*[2] As before, in order to have lines of direction to the *P.P.* preserve the same relation to the vanishing-trace of their plane on the paper that they do in space, it is necessary to rotate the *S.P.* about the trace of the plane of the lines considered, into the *P.P.* In this problem *S.P.*[1] *C.V.* is the radius of rotation, *C.V.* its center, and *S.P.*[2] the rotated position, i.e., on a perpendicular to the vanishing-trace of the plane of the square. From here the process is the same as in the preceding problem. *AB* and

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* Because it contains lines perpendicular to the picture-plane.

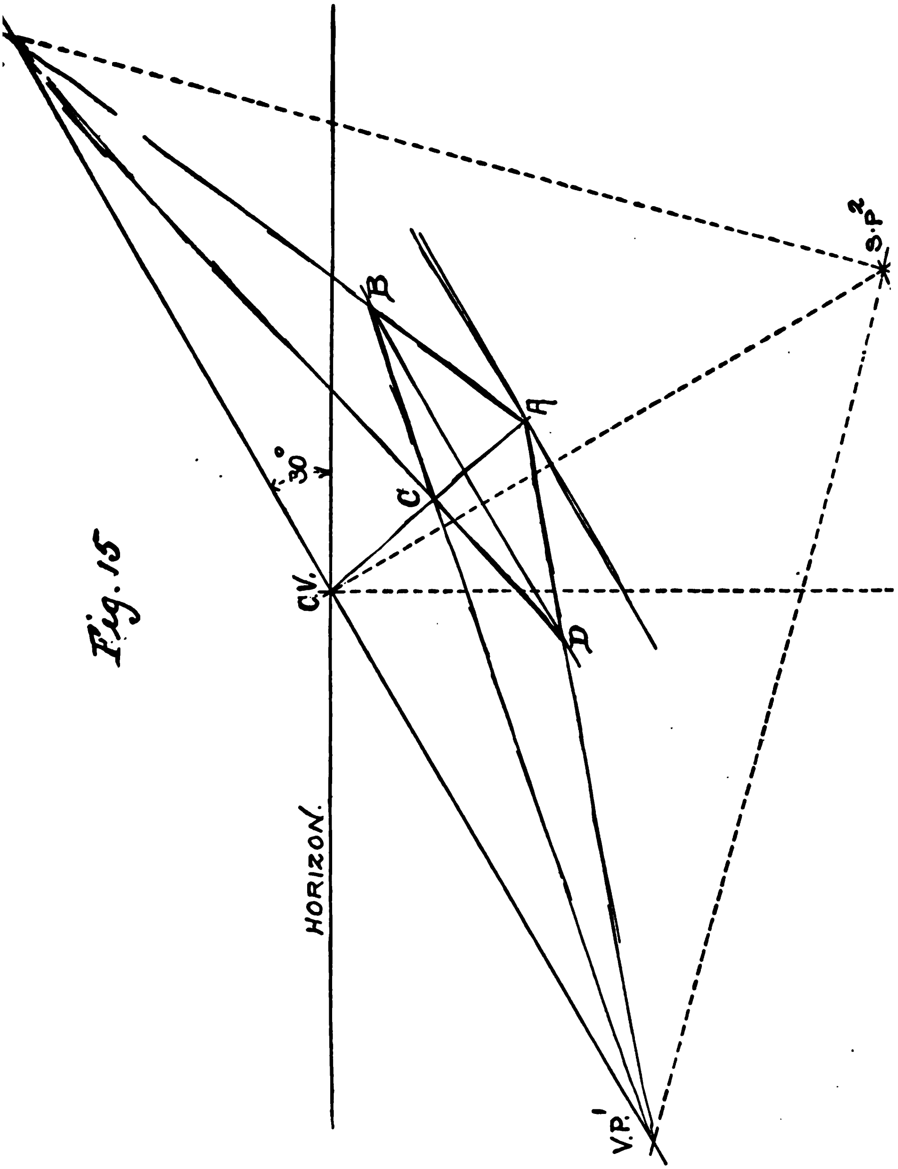

Fig. 15

AD will be nearly equally inclined to a line which is parallel to the vanishing trace of the plane of the square, one diagonal of the square being parallel to this line and one vanishing in *C.V.*

It would be a simple matter to apply the method of section 38 to this case. A line would be first sketched at the required angle to the horizontal, which might be taken to represent the line of intersection of the plane of the square and the P.P., and passing through the nearest corner of the square.

42. Problem 5. A Square Plane Oblique to Both the P.P. and the Horizontal Plane.—Sketch a square with edges equally inclined to the horizontal plane, but whose plane is oblique to both the horizontal plane and the P.P.

First it is necessary to decide upon the conditions to be used for determining the position of the square; these can be established in a variety of ways, but one of the simplest is to locate the line of intersection of the plane of the square and the horizontal plane (see Fig. 16). Assume this line to make an angle of 60° with the P.P. inclined towards the left; in addition assume the angle which the vanishing-trace of the plane of the square makes with the horizon to be 45°, facing towards the right.*

Having located a horizon and an *S.P.*[1] then, by the method of section 33, *V.P.*[1] can be found, and through this point can be drawn the vanishing-trace of the plane of the square at the required angle to the horizon. In order to find the vanishing-points of the receding edges of the square it is necessary first to rotate the *S.P.* as before into the *P.P.*, and into such a position that the lines of direction to the *P.P* and parallel to the edges of the square will hold the same relation with the trace of the plane of the square after the rotation that they did in space. The rotation must be in a plane perpendicular to the vanishing-trace of the plane of the square and passing through *C.V.* This requires, as will be seen by reference to Fig. 16, a radius of rotation with *V.P.*² as a center, which is the

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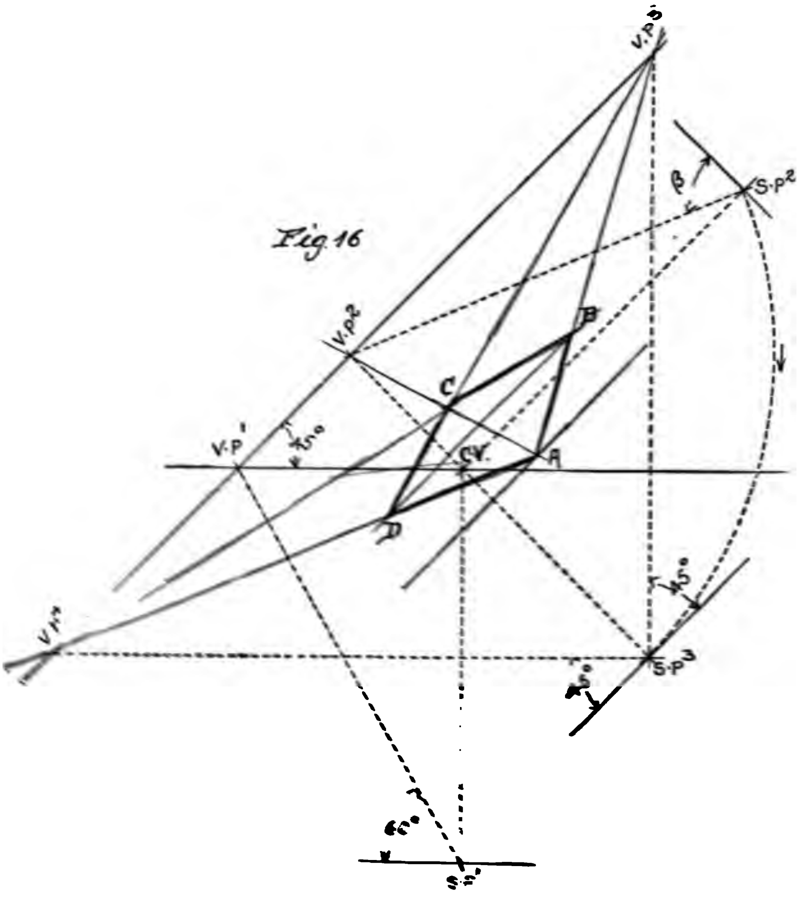
* It must be understood that the angle between these two lines does not indicate the angle which the plane of the square makes with the horizontal plane. The angle between two planes is always measured by means of lines cut out by a plane perpendicular to their line of intersection, which the picture-plane is not in this case. It was true, however, in the case of problem 4.



hypothenuse of a triangle of which the distance the observer is from the P.P., namely, S.P.¹ C.V., is one side, and the perpendicular distance that C.V. is from the trace of the square, C.V. V.P.², is the other; this triangle is for convenience shown as rotated into the plane of the picture upon its side *C.V. V.P.*², and *V.P.*² *S.P.*² is the hypothenuse. Therefore from *V.P.*², and perpendicular to the trace of the plane, the distance *V.P.*² *S.P.*³ is laid off equal in length to the hypothenuse; from *S.P.*³ the lines of direction can now be drawn, each making 45° with the vanishing-trace of the square, or they can be measured from a line parallel to the vanishing-trace of the square. The remainder of the problem is similar to those preceding, except that it is seen that one diagonal of the square vanishes in *V.P.*²; this is because it may be conceived to lie in a plane which is perpendicular to the *P.P.* A line through *V.F.*², *C.V.*, and *S.P.*³ would be the trace of such a plane.*

43. Summing Up.—In conclusion it may be said that these five problems illustrate graphically all the possible positions which straight lines can occupy. The vanishing-point of any line can be found if the direction of the line is known, and of course its direction involves referring it to other lines or planes. The vanishing-trace of any plane can be likewise found if its relation to other lines or planes is known, or if two of its points can be located. In these problems it is seen, also, that to locate definitely the vanishing points of the receding edges of the square, a rotation of the S.P. into the P.P. is necessary in each case, the rotation always being such that the lines of direction from the S.P.¹ to the vanishing-trace of the plane considered will bear the same relation to that trace that they did in space; the required rotation, moreover, is always in a plane perpendicular to the P.P. and to the vanishing-trace. The application of the principles enunciated can be made more varied by changing the conditions given. The draftsman, however, is rarely called upon to locate his station-point and work out the vanishing-points of lines; their direction, if it gives him concern, is determined by their relations to one another, such as parallelism, angularity, etc., so that he need not definitely determine where the points are.

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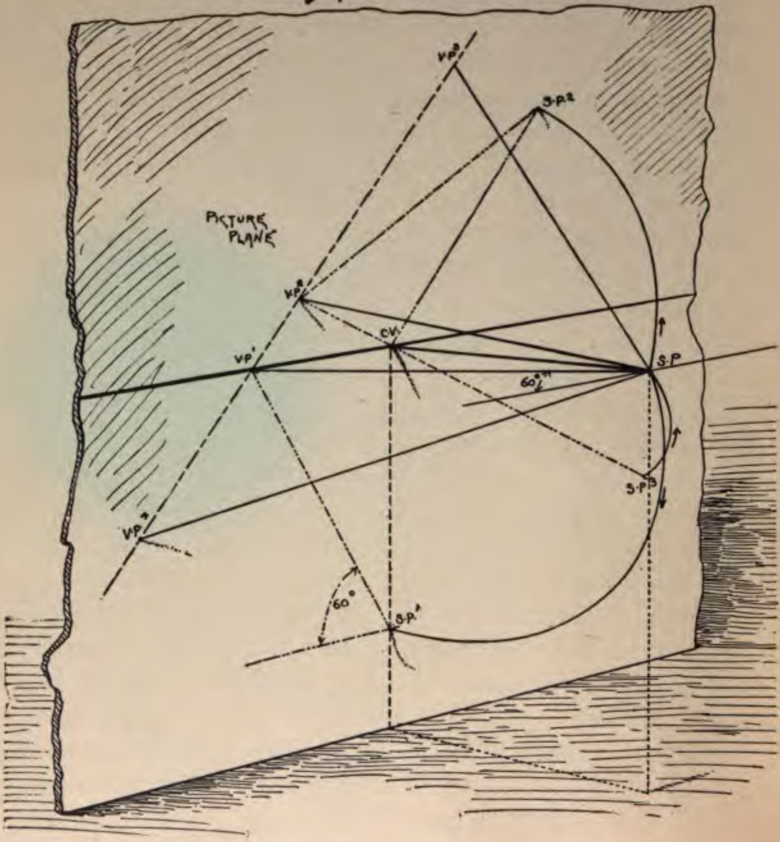
* The angle β in the figure indicates the angle between the plane of the square and the *P. P.*, and of course it might have been established if desired in the beginning as a condition of the problem.

In the hope that it may make more clear any ambiguity which the mixture of lines on the P.P. in the last problem may give rise to, in the student's mind, Fig. 17 is introduced showing a picture of the picture-plane and the various lines of direction in space. The arrows show the directions and the arcs of rotation, and the differing character of the broken lines will help to separate them into classes; that is, the imaginary

Fig. 17.



lines of direction in space are solid, and the lines on the picture-plane are dot and dash, with the exception of the rotated 'line of sight,' which is dotted.

44. **Miscellaneous Problems.**—Following are a few problems embodying the foregoing principles, which it is the intention should be done by sketch methods.

Problem 1. Sketch two horizontal squares, corners touching, edges equally inclined to the P.P. Next sketch three vertical squares, perpendicular to the P.P., and edges equally inclined to it, their lowest corners touching the points in the first squares lying on the diagonals which are continuous and parallel to the P.P.

Problem 2. Sketch two squares whose planes are perpendicular to each other and to the P.P., and edges equally inclined to it and cutting each other so that their diagonals coincide. The plane of one is at 30° to the horizontal plane, and that of the other at 60° to it. Either sketch one of the squares first and from it derive the perspective conditions or establish them in the beginning.

Problem 3. Sketch a square whose edges are equally inclined to the P.P. and whose plane makes about 20° with the horizontal plane and 70° with the P.P., i.e., whose vanishing- and initial-traces are horizontal lines in the picture. Establish the perspective conditions by locating the S.P.[1] and horizon first.

CHAPTER IV.

THE MEASUREMENT OF LINES.

45. Measurements in Perspective are Relative.—Before taking up the study of the perspective measurements of lines in detail, it will be well to consider the subject of measurements in general. Despite the fact that in works on perspective quite considerable emphasis is usually laid on measuring lines by a variety of devices, the value of these processes in sketch methods can easily be overestimated. Perspective measurements are purely relative, because objects change their size according to their distance from the observer and the picture-plane; no one can tell unless he applies geometrical and perspective tests whether a certain object drawn in a picture is just the right size or not. In fact some familiar object must be present before even a rough estimate of sizes can be made; this is why a human figure is so invariably introduced into pictures of buildings, or a human hand, for example, into illustrations of small subjects in our trade journals. The term 'drawing to scale,' then, has not the same significance in perspective that it has in other mechanical drawing, and particularly in sketching it is found that measurements can be better estimated than scaled, and if estimated with a reasonable degree of accuracy will be as correct as can be expected.

In order, however, to be able to apply them when desired, some clear idea of the principles underlying the measurements of lines should be had, and for this purpose it is proposed to go very briefly over the ground.

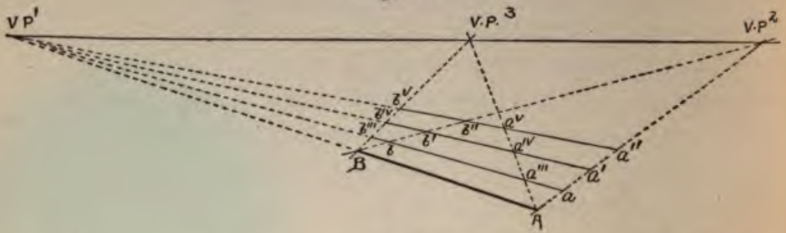
46. The Measurement of Lines Geometrically.—As plane geometry is at the basis of perspective, it will be well to consider measurements as made geometrically; these measurements may really be divided into two classes: first, by means of intersecting parallel lines, this depends upon the principle that parallels between parallels are equal; second, by means of the compass, that is, when two separated lines can be placed so that they intersect, one can be measured from the other by using a compass with the vertex of the angle formed between the lines as a center. In place of the compass the dividers would be set to the length of one line and transferred to the other. In the case of measuring with the compass, if the initial and transferred measurements were connected by a line, an isosceles triangle would be formed, and here it may be noted that the base of this triangle, forming the chord of the arc made by the compass, might be drawn in place of the compass arc and give the measurement as well, provided its angle with the sides could easily be found. This is possible in perspective, as will be seen presently, and is the basis of many of the measurements used.

47. Application to Perspective.—In perspective, measurements can be similarly divided into two classes: first, by the principle of parallels between parallels being equal; second, by the method of triangles. The first is probably the most frequently used by the draftsman, for reasons which will be obvious later.

As an introduction to the study of perspective measurements it is necessary at the outset to keep clearly in mind the fact that whenever the pictures of lines are drawn, whether they be measuring lines or measured lines, for each and every one an imaginary line of direction is drawn parallel to it through the station-point. Hence if two intersecting lines are measured equal in length by means of a third, which is the chord of the arc previously mentioned, this third completes with them an isosceles triangle. If put into perspective, imaginary lines of direction are drawn starting from the station-point and parallel respectively to the sides of this triangle; these latter lines do not form a triangle which is similar to the first, radiating as they do from a point, but for the purpose of clear comprehension of the analogy they may be conceived to be so placed that they do; this assumed position moreover is permissible because one of the lines, the line which forms the scale of measures, must be parallel to the picture-plane, and for it one on the picture-plane may at once be substituted.

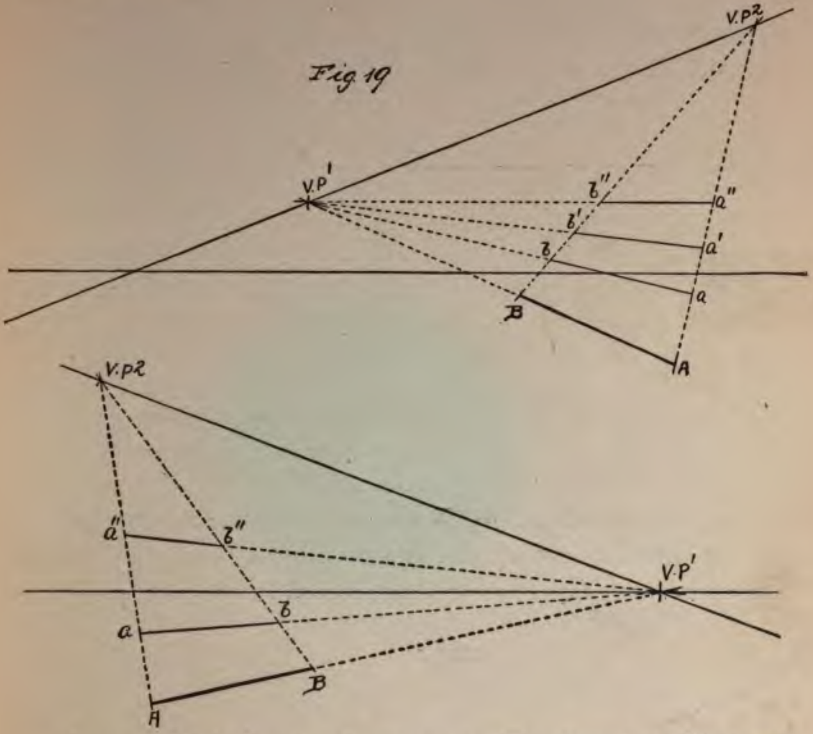
48. Measurements by Means of Parallels.—The most general case of measuring by parallels may be exemplified in Fig. 18. Here is shown a series of horizontal indefinite lines parallel to AB, which is taken of definite length.

Fig 18.



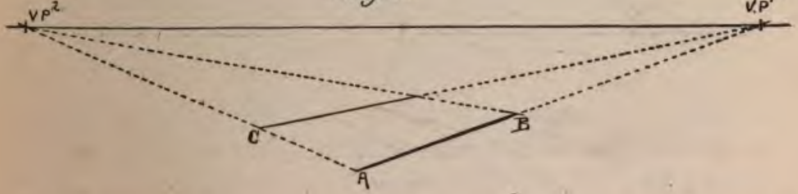
Measurements on these indefinite lines equal in length to AB can be made starting at any point, namely, from A draw a line to meet the vanishing-trace of the plane of the lines in any point $V.P.^2$, which will be the vanishing-point for that line, then another line from B drawn to this point will give $ab = a'b' = a''b'' = a'''b'''$, all equal to AB. Any other vanishing-point on the trace, such as $V.P.^3$, could have been taken instead, giving $a'''b''' = a^{\text{iv}}b^{\text{iv}} = ab$, etc. Of course the vanishing-points $V.P.^2$ and $V.P.^3$ must be taken in the plane of the lines or they will not represent the vanishing-points of lines intersecting the indefinite parallels. If this is borne in mind, then it can be readily seen that the process will apply to lines lying in any other than horizontal planes. Fig. 19 is introduced to show this. In the first case the lines to be measured lie in a plane which is inclined to the horizontal plane, and the lines themselves are not horizontal; in the second case the lines are horizontal, but the plane which may be passed through and containing them all is not. It is necessary to have the trace of the plane of the lines in each case before the measurements can be affected.

Fig 19



A case which is not quite so general is where two lines are in a known horizontal plane (see Fig. 20), and one of the lines

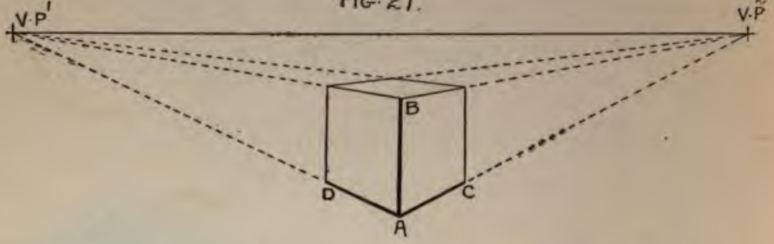
Fig. 20



to be measured has one extremity fixed at *C*. A line through *A* and *C* will give *V.P.*², the vanishing-point of the measuring parallels.

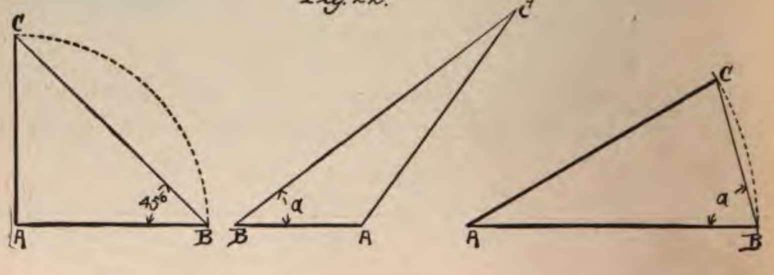
A graphical application of the method of parallels is shown in Fig. 21. Here, if the three darkened edges, *AD*, *AC*, and *AB*, can be established, the remainder of a cubical figure can be completed by the drawing of parallel lines. Any rectangular solid can be constructed in a similar manner. The draftsman sketching it would possibly use these preliminary lines. For further illustration of the measurements by means of parallels study the preceding figures 1, 4, etc.

FIG. 21.



49. Measurements by Means of Diagonals of Squares.—In geometry, as before stated, two intersecting lines can be measured equal in length by the compass (see Fig. 22). In the first, *AC* stands at 90° to *AB*, and the chord of the arc, or the third side of the triangle formed, makes with the other sides an angle of 45°. To construct a 45° triangle, then, would be the quickest way to effect this measurement. To make it in the other two cases with the straight-edge, as also before stated, would require a knowledge of the value of the angle *α*.

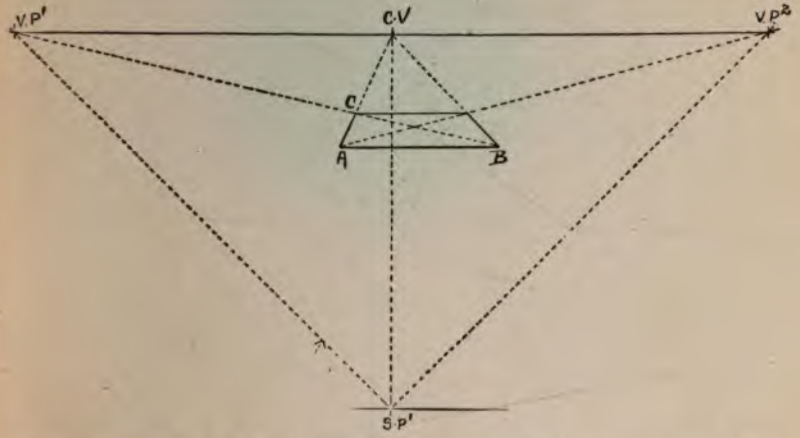
Fig. 22.



In perspective it is far easier to deal with straight lines, hence the chords would be used.

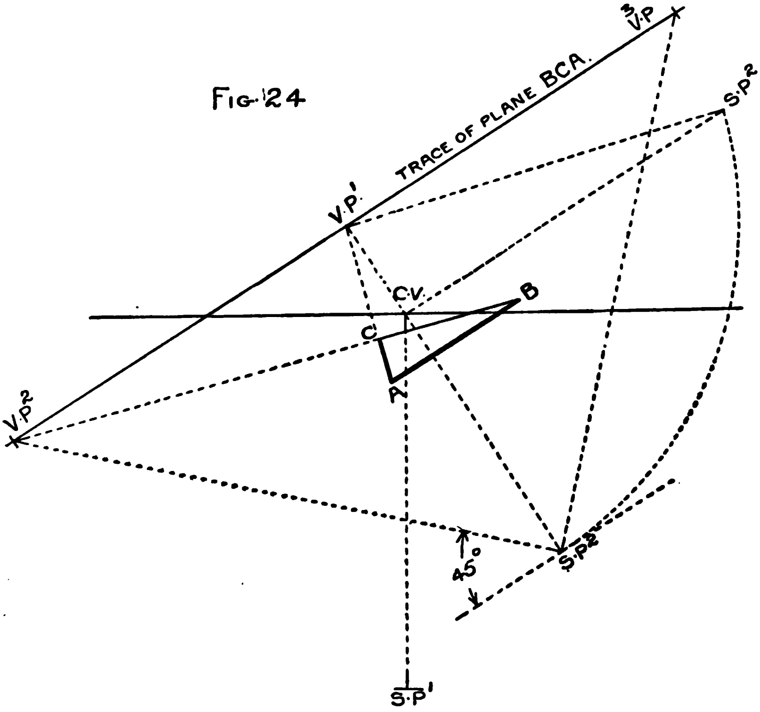
Consider a few moments the first case, namely, the 45° triangle. It is seen that this triangle forms one-half of a square, and in treatises on perspective, where lines so situated relative to one another are discussed, the method is given a separate division, as that of '*diagonals of squares.*' The application of this method in perspective appears most frequently when the line measured from is parallel to the picture-plane, and the line measured lies in a plane perpendicular to the picture-plane. This is exemplified in Fig. 23, where the plane of both lines is assumed for simplicity to be horizontal. After the $S.P.^1$, the horizon and $C.V.$ are assumed, the vanishing-point of the hypothenuse of the right-angled triangle ABC is found at $V.P.^1$ by section 33. $V.P.^1$, in such cases as this, is frequently called the '*distance-point,*' because it is a distance from $C.V.$ equal to the distance the observer is from the plane of the picture. Any line which is parallel to AC can be similarly measured by a line vanishing in $V.P.^1$ Note also that we have three imaginary lines of direction $V.P.^1\ C.V.$, $C.V.\ S.P.^1$, and $S.P.^1\ V.P.^1$, the latter two rotated into the

Fig. 23.



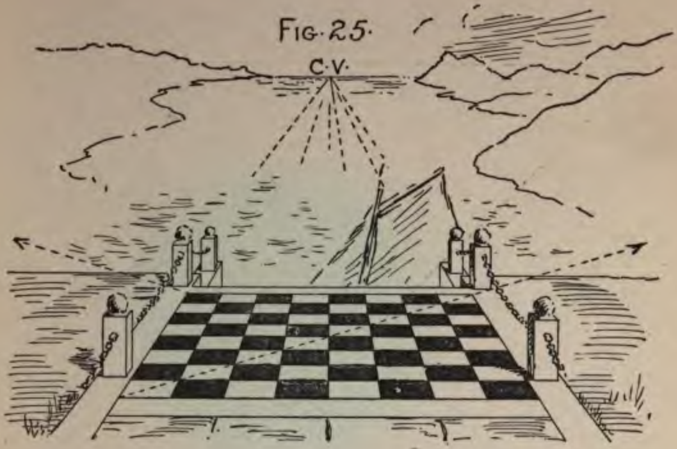
P.P., all parallel when laid out in their original position to the three sides respectively of the triangle *ABC*. It can now be seen by a slight investigation why the chord of the arc represented in Fig. 22 is used to measure with more conveniently than the arc. The line of direction *S.P.*^{1} *V.P.*^{1} is originally parallel to *BC*, and itself is the base of an isosceles triangle formed by lines of direction, just as *BC* is the base of the triangle *ABC*. The line of direction *S.P.*^{1} *V.P.*^{1} can always be found independent of its angle with the picture-plane, because it measures *V.P.*^{1} *C.V.* equal to *S.P.*^{1} *CV.* Fig. 24 is introduced to show the construction when the plane of the lines is oblique.

FIG. 24



An interesting application of measurements by parallels and by the diagonals of squares is shown in Fig. 25. It is a pavement laid out in square tiles. The width of each tile is shown by parallels running perpendicular to the picture-plane; their depth in each case is located by the point at which the diagonal of a square tile cuts the side. From the construction it is evident that time has been saved by using one diagonal of a large square instead of the separate diagonals of the smaller ones.

FIG. 25.

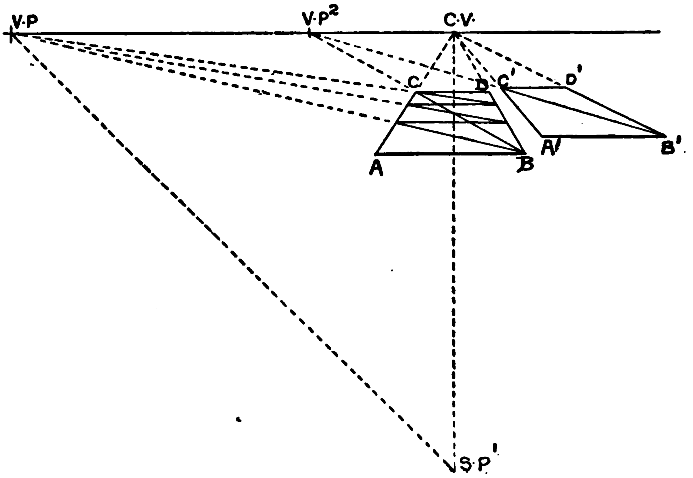


50. Measurements by Means of Diagonals of Rectangles.

—It may be well to note before proceeding with the final class of measurements that measurements by means of diagonals can be applied with equal propriety to any rectangular figure, though it is not of considerable utility in sketching. Fig. 26 illustrates this in the case of a horizontal rectangle, *ABCD*. It is seen that *AC* represents by construction a line three times as long as *AB*, if the diagonal of the rectangle thus formed is produced to its vanishing-point *V.P.*[2]; then it, in turn, can be used to measure any other two lines *A′C′* and *A′B′* similarly situated and of the same relation as to size. It is instructive to regard that the distance *AC* as obtained by transferring the scale *AB* three times from *AB* into the picture, the three lines parallel to *AB* being the scale in each case.

51. Measurements by Means of Isosceles Triangles.—The use of diagonals of squares is one special case of measuring by the isosceles triangle and generally is accorded separate treatment. It will be recalled that in section 49 the three imaginary lines of direction were originally drawn parallel to the three sides respectively of the triangle represented in the picture, and that *BC* was its base. If any two intersecting lines are given, one of which is parallel to the picture-plane, and upon which consequently a scale can be laid off, it is possible by finding the vanishing-point of the base of an isosceles triangle of which these two lines are the sides to make measurements on the one line equal to those on the other. This, in brief, is the statement of the process. To make it more clear the following problem is added.

Fig. 26

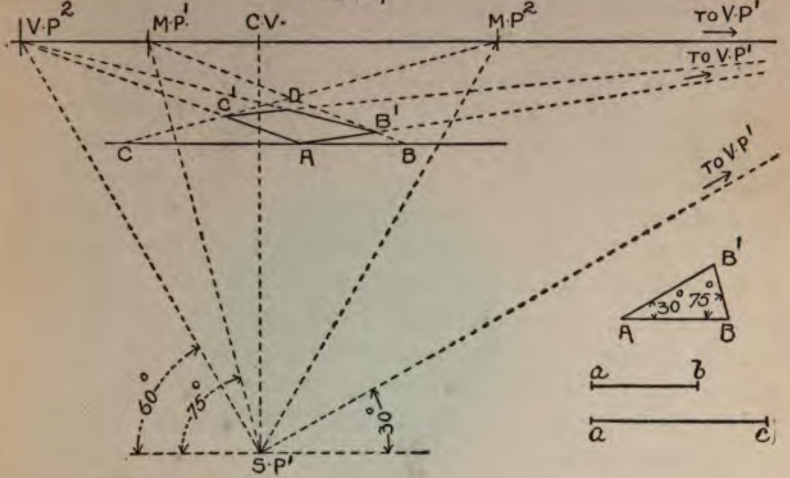


52. Problem in Measurement by Triangles.—Sketch a rectangular plane two units long by one unit wide, lying horizontal, below the level of the eye, its short edge at 30°, and its long edge at 60° to the P.P.

Sketch a horizon, a station-point *S.P.*[1], and a *C.V.* (see

Fig. 27), and find by section 33 the vanishing-points of the edges of the rectangle. At *A*, any convenient point near the *C.V.*, sketch indefinite lines to the vanishing-points *V.P.*^{1} and *V.P.*^{2}; now assume *ab* and *ac* as the lengths of the adjacent sides of the rectangle, one twice the size of the other; if *ab* is laid off on a line in the plane of the rectangle, as at *AB*, then it is necessary to find the vanishing-point of such a line which when drawn will measure the short receding edge equal in length to *AB*. This line is obviously the base of an isosceles triangle. By referring again to section 49, a suggestion can be found for finding the vanishing-point. In Fig. 23, *V.P.*^{1} *C.V.* was made equal to *S.P.*^{1} *C.V.* If upon this suggestion in Fig. 27 a distance is laid off on the vanishing-trace of the plane of the rectangle from *V.P.*^{1} which is equal to the length *V.P.*^{1} *S.P.*^{1}, then *M.P.*^{1} will be the vanishing-point required. The point is found more easily this way than if a calculation had been made of the angle which the base made with the sides, and this angle constructed at *S.P.*^{1} A line from *B* to *M.P.*^{1} will give *AB'* as the perspective length of the short side of the rectangle.

Fig. 27



The point $M.P.^1$ is called for convenience the measuring-point, and the 'one' over the P is to associate it with $V.P.^1$, so that it can be found easily in a drawing of many lines as the measuring-point to be used in measuring all lines which go to $V.P^1$. To verify the construction of the triangle ABB' it is seen that there are three imaginary lines of direction which in their original position in space are parallel to the three sides of the triangle ABB', namely, $S.P.^1\ V.P.^1$ is parallel to AB', $C.V.\ V.P.^1$ is parallel to AB, and lastly $S.P.^1\ M.P.^1$, the base of the larger triangle, is parallel to BB', the base of the smaller triangle. To measure the adjacent side of the rectangle recourse is had to a similar process; $V.P.^2\ M.P.^2$ is made equal in length to $V.P.^2\ S.P.^1$, so that C' is located. From here parallels complete the figure. It will be seen in a later chapter how the angles BAB' and CAC' can be estimated closely enough for all practical purposes, so that the exact measuring-points can be dispensed with.

53. Rule for Finding the Base of the Isosceles Triangle.—The above problem is typical of all the possible cases which may arise of measuring by triangles; of course the obliquity of the plane of the triangle formed may obscure the real conditions, but a rule can be formulated for finding the measuring-point for any system of parallel lines applicable to all cases, namely: *Lay off on the vanishing-trace of the plane of the lines to be measured a distance from the vanishing-point of the lines equal to the distance the station-point is from their vanishing-point; this distance will give the measuring-point required.*

54. Additional Illustrative Example.—Fig. 28 is given as illustrating the measurements by triangles and parallels when the planes of the lines are oblique to the P.P. If these planes are kept clearly in mind together with the principles just enumerated, it should not be difficult to understand the figure. However, to help the student to an understanding of the steps the following very brief description is given.

The subject is a cube standing on one edge, this edge

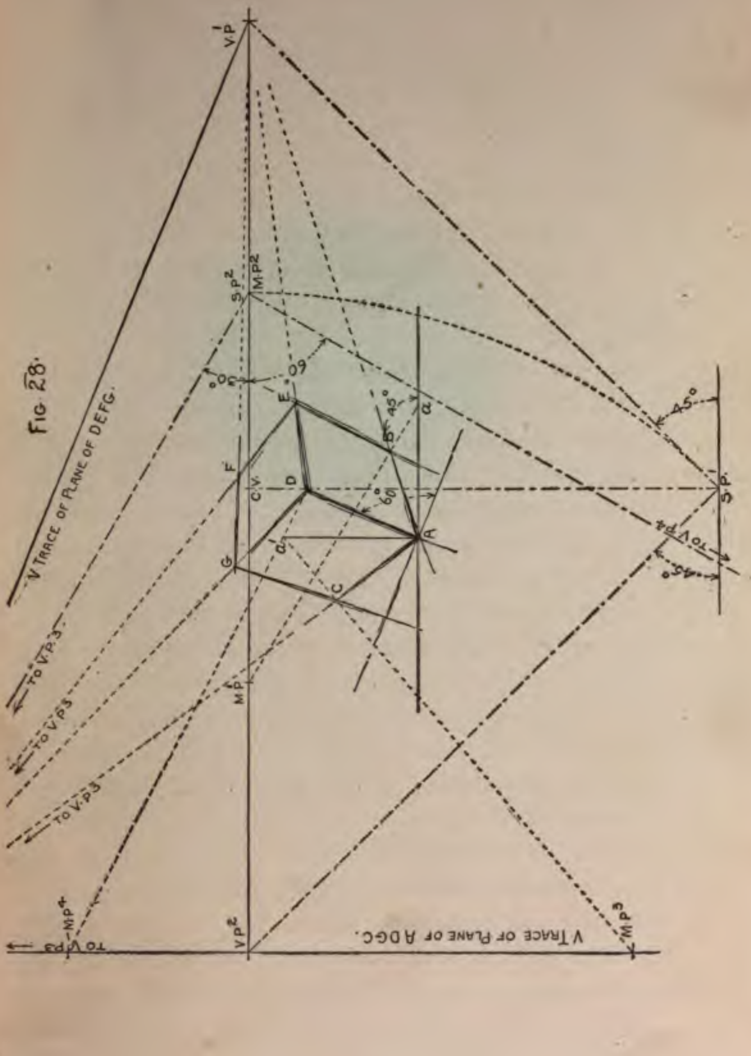


FIG. 28.

V.P¹
S.P²
M.P2
60°
60°
E
45°
a
B
F
D
C.V.
60°
A
G
C
a
M.P
V TRACE OF PLANE OF DEFG.
TO V.P.3
TO V.P3
TO V.P3
MP⁴
TO VP3
VP²
V TRACE OF PLANE OF ADGC.
M.P³
S.P.
TO V.P4
45°
45°

making an angle of 45° with the P.P. and an inclined face an angle of 60° with the horizontal plane.

The edge *AB* on the ground is first sketched to its proper vanishing-point *V.P.*[1] and its length either measured or assumed. If it is assumed, then its true length to scale *Aa* must be found by drawing through *B* a line from *M.P.*[1] The plane of *ACGD* is next located, its vanishing-trace being vertical. The vanishing-points of the edges lying in this plane are found by a rotation of the observer about its vanishing-trace, with *S.P.*[1] *V.P.*[2] as a radius, namely, to *S.P.*[2] From here the angles are constructed and the measuring-points found the same as if the plane of the lines was horizontal. *Aa'*, the length of an edge, is in the same plane as the face of the cube *ACDG*, and is used in the same manner to get *Ac*, similarly with the edge *AD*.

55. Observations upon Some Minor Details.—In what has just been said no mention was made of exactly what scale was used by which to make various measurements, nor was any made of whether the object were standing on the ground or not. These matters are so often discussed that it seems worth while to give reasons why they have here been omitted. In making a sketch it is of little concern either to the draftsman or observer just where the picture-plane is situated relative to the objects drawn. Rarely can it be assumed as touching that object nearest the observer, for that part of the object would necessarily be its true size, and it is comparatively seldom that drawings are made life-size. Some part or line of the nearest object would be established arbitrarily, and this would furnish the scale for everything else. Any line taken in the same vertical plane * with this scale can be divided with the same unit of measurement; this only changes as the scale is transferred into the picture as exemplified in the successive horizontal lines of Fig. 26. If the student were drawing these problems on paper, it would be perhaps best to use sketch methods as far as possible and take such a unit for the first measurements as would promise to give the most pleasing result. The direction has been several times given to start relatively near the center of vision so as to insure this. The limitations as to position which should be observed will be dwelt upon later.

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* Of course the picture-plane is not necessarily vertical, though it is generally so drawn.

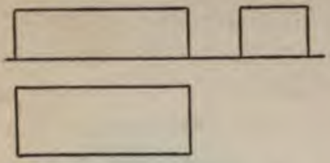
Again, in these simple problems it has been of little importance whether the objects were standing on the ground or on some horizontal plane. It is always an easy matter to give the effect by a few shade-lines of a horizontal plane as a base for the object. It frequently happens in discussions on perspective that a ground-line is spoken of; this line is the intersection of the picture-plane with the horizontal plane: measurements are laid out on this, and if the object drawn does not touch the picture-plane, then they are transferred into the picture by the method described. But these first measurements, it would seem, are a superfluity and might readily be dispensed with, as they in no sense help to suggest the true sizes after they are applied to the subject; the familiarity of the observer with the latter or its relation to other objects will carry a true enough impression.

56. Problems in Measurements.—That the student may be enabled to fix the principles of measurements in mind the following problems are given. It is to be understood that the minimum of constructive work should be used, enough to locate the vanishing-points correctly, the remainder to be by sketch methods such as were used in the demonstrations of the problems in sections 37 to 42 inclusive; also a horizon should be located and the observer placed as far from the horizon as will permit of the constructive work coming within the limits of the space available for the drawing.

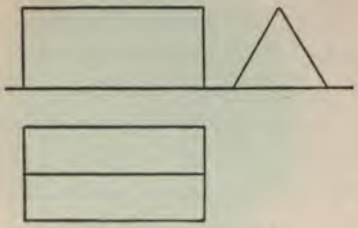
Problem 1. Sketch a railway track, the rails making an angle of 60° with the P.P., and show the ties, a distance apart approximately equal to one-fourth the distance between the rails. Assume the unit of measures.

Problem 2. Sketch the same, the rails at 45° to the P.P., and add a row of telegraph-poles a distance from the track of twice the width between the rails, and space the poles five times this latter distance apart. Use the method of diagonals of rectangles for spacing the telegraph-poles.

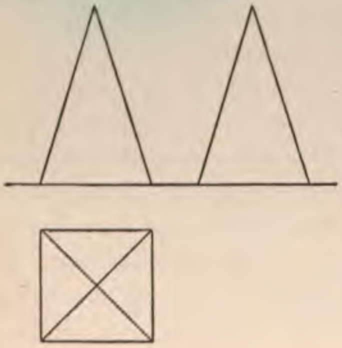
Problem 3. Sketch a square-tiled pavement similar to that of Fig. 23, but make the following modification: place the tiles so that their four edges are all equally inclined to the picture-plane.



Problem 4. Sketch the following block, of the size shown, standing on its large rectangular face, the vertical faces at angles of 40° and 50°, respectively, to the P.P.



Problem 5. Sketch the triangular prism of size shown, standing similarly, the edges of its base on which it is standing equally inclined to the P.P.



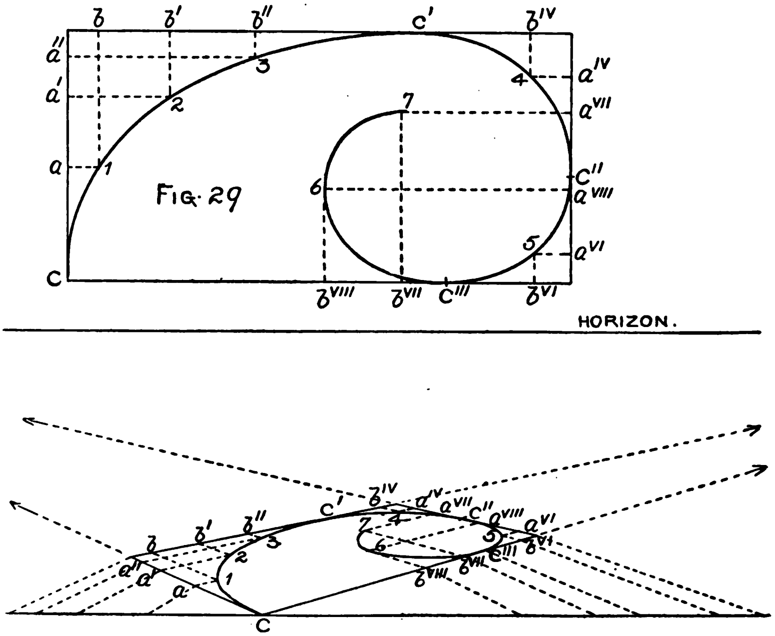
Problem 6. Sketch the following square pyramid, standing on its square base, the edges inclined at 30° and 60° to the P.P.

CHAPTER V.

THE TREATMENT OF CURVES.

57. The Method of Circumscribing Polygons.—A curve, as commonly understood, is a line which has changing direction so that it cannot, like a straight line, be located perspectively by laying out its direction at the station-point or by fixing two points in it. If the curve is a plane curve, however, and its plane is parallel to the picture-plane, then it presents no perspective changes in appearance except a reduction in size as it is moved away from the observer. In general, the method adopted for dealing with curves is to refer them to straight lines, more often polygonal figures surrounding the curves; that is, the various points of a curve are located by measuring the distances of these points from two or more straight lines or from the sides of the circumscribing polygon, which latter figure can be very easily located perspectively by the principles already discussed.

58. Illustrative Example.—This can be exemplified by reference to Fig. 29, showing an irregular curve circumscribed with a parallelogram and the perspective of both. The parallelogram taken is that which most conveniently bounds the curve, the latter being tangent to it at certain points, C, C', C'', and C'''. As many other points in between, 1, 2, 3, etc., are taken as seem desirable, and these referred to the sides of the parallelogram by the dotted parallels shown. In the perspective these parallels are easily drawn and their corresponding intersections give the same points perspectively; through them a smooth curve can be drawn. It is usually most convenient also to so construct the circumscribing parallelogram that two of its edges are parallel to the picture-plane. This makes it possible to transfer measurements to the sides directly or by the method of diagonals of squares, instead of the position shown above, in which the measurements are transferred by the method of isosceles triangles. Curves of

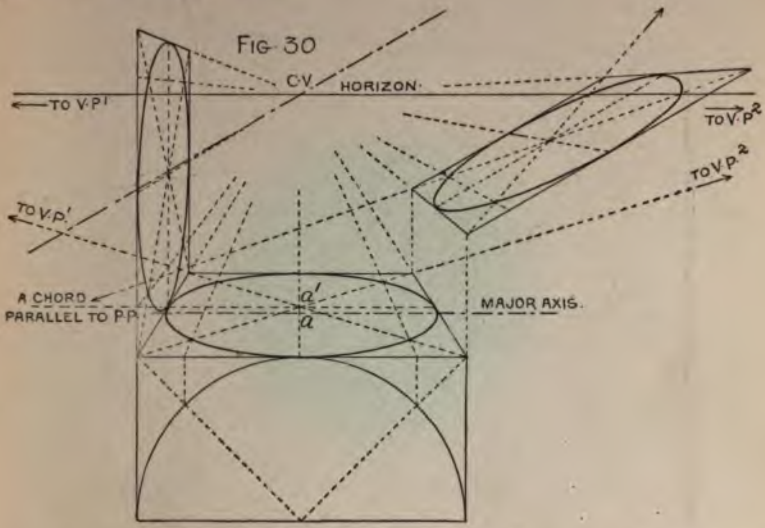


innumerable shapes may thus be worked out, the same method applying in all.

59. **The Circle in Perspective.**—The circle, in whole or in part, is probably the most common of curves, and hence deserves to be studied somewhat in detail. The square is the most natural circumscribing figure for the circle, and, in the absence of any other limitations, one with edges parallel to the picture-plane is the easiest of construction. The points to be taken also which first suggest themselves are the points of tangency to the sides of the square, and those in which the

circle intersects its diagonals, and as many taken in between these as seem necessary. Fig. 30 serves to illustrate this. It is the picture of three equal circles, one in a horizontal plane, one in a vertical plane, and one in an oblique plane. The general appearance of the three is seen to be the same; in fact they are ellipses. The construction used to find the various points is also indicated.

FIG. 30



The fact, noticed upon careful inspection, that the axes of the ellipses are not coincident with the diagonals or diameters of the squares will be referred to a little later in detail.

60. The Circle and Conic Sections.—In order to more fully appreciate the possible forms which the picture of the circle may assume, a study of the perspective conditions interpreted in the light of conic sections will be profitable. Fig. 31 shows the relation of an observer to a circle which is lying on the ground in front of him. Imaginary lines of direction from his eye to various points on the circumference of the circle, such as would be conceived to be drawn to intersect a picture-plane placed vertically, form an oblique cone (with a circular base). Now if the observer is wholly outside of the circle, as

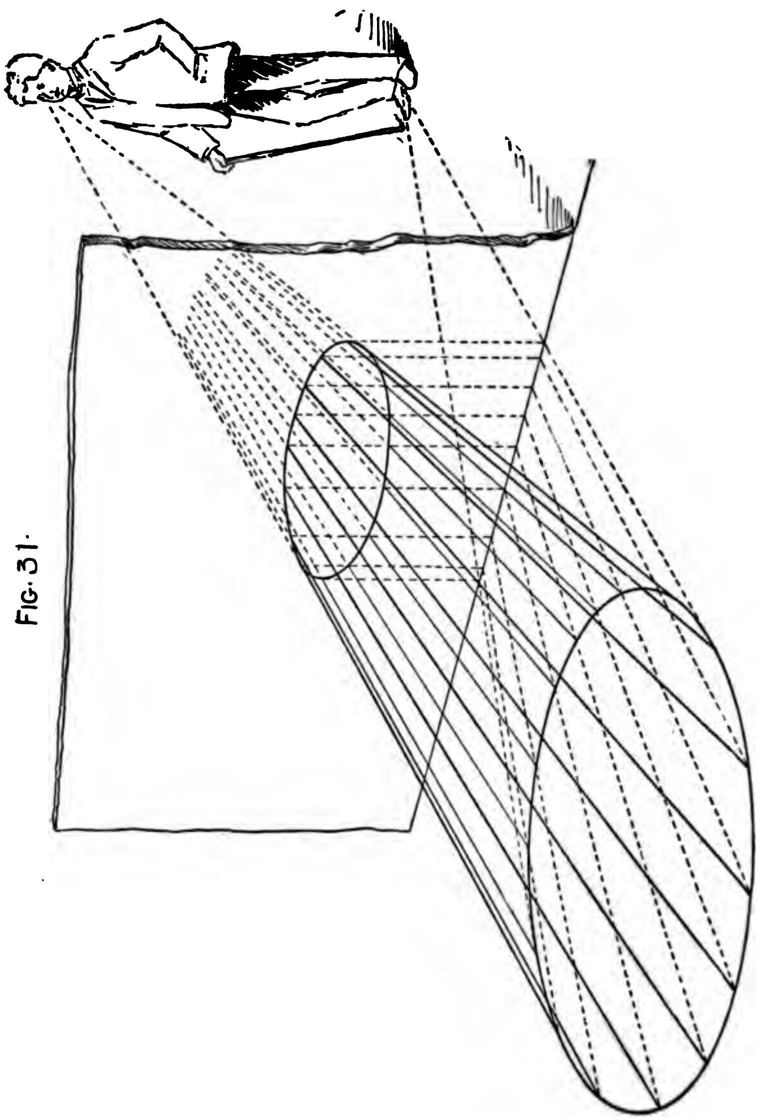


FIG. 31.

in the figure, then, from descriptive geometry, it can be proven that the vertically interposed picture-plane will cut the cone in a plane section bounded by an ellipse.* If the observer move towards the circle until he is standing on the edge of it, then the vertically interposed picture-plane will be parallel to an element of the conical surface and the section so formed will, also from descriptive geometry, be bounded by a parabola. If he move still further, until he stands upon the center, then the section formed will be bounded by a hyperbola. The last two forms are not of general use to the draftsman; they rarely occur in practice.

61. Possible Forms of the Circle in Perspective.—If the observer is wholly outside of the circle when he is making his sketch, then the circular plane can have one of three forms, depending upon his position: (1) a straight line, (2) a circle, (3) an ellipse.

If the plane of the circle, extended, passes through the observer's eye or station-point, that is, if he is looking tangent to its plane, then the resulting picture is a straight line. If the plane of the circle is parallel to the picture-plane, then the result will be a circle. In any other than these positions the picture will be an ellipse, and this last is by far the most common form to be met with in practice.

62. Two Facts of Importance in the Picture of the Circle.—In section 59 it was remarked that the axes of the ellipse were not coincident with the diagonals or diameters of the circumscribing square; the position of the axes is determined by the nature of the conic section formed by the picture-plane cutting the cone before described in section 60. A very simple demonstration will suffice to show graphically that they are not related to each other. About any circle, lying in a horizontal position, let us say, it is possible to circumscribe squares whose edges are variously inclined to the picture-plane (see Fig. 32). The pictures of these squares are severally different, while the circle remains unaffected. Or again:

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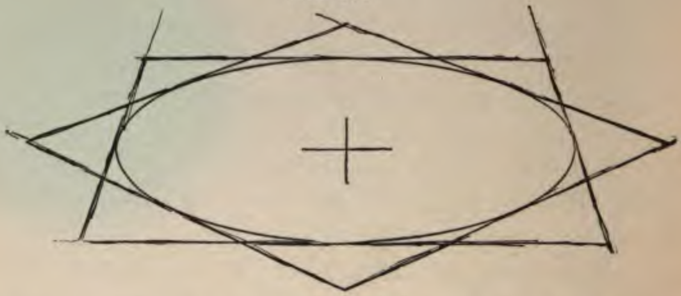
* The proof of this is not within the province of this treatise.

suppose the circle, together with its circumscribing square, to be revolved about a vertical axis through its center: the successive positions of both will result in different forms to the square, but the circle is not changed at all. This fact is almost self-evident, but unfortunately it is not always fully appreciated.

Another fact of importance to be noted is in the nature of a corollary to the above. It can be seen in Fig. 30 that *a* is the center of the ellipse, and that *a′* is the picture of the center of the circle together with that of the circumscribing square; also, the major axis terminates in the extreme right- and left-hand part of the curve, and does not touch the middle of the opposite sides of the square. The distance between the above two points is slight, but at the same time always present. Within certain limits it grows less as the plane of the circle is viewed at a more and more acute angle, and it becomes greater, within limits, when viewed more and more perpendicularly. As will be seen a little later, for the purposes of ordinary sketching this difference between the two can in general be ignored and both assumed as identical. The assumption, though not strictly true* in a geometrical sense, will not lead to any appreciable error in the drawing, and will often save labor in interpreting a form which gives more than common difficulty to the draftsman.

FIG. 32.



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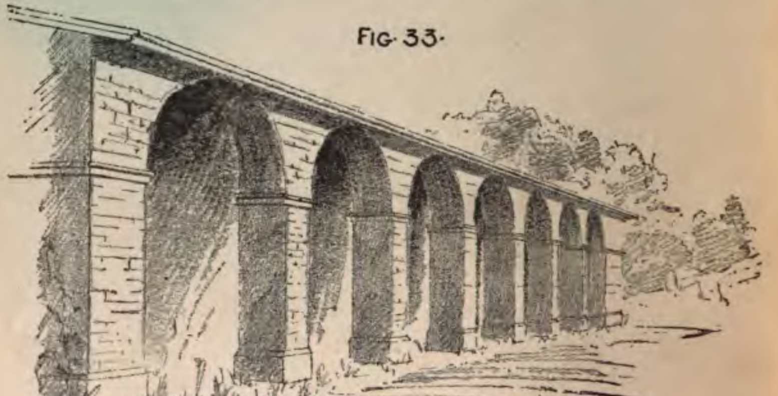
* True as looked at from the standpoint of descriptive geometry or conical projection.

63. Compromises made by Errors Introduced into Pictures.—Somewhat in advance of the general plan it becomes necessary, for a fuller understanding of the treatment of the circle, to consider briefly a few of the difficulties in the way of adhering strictly to true perspective processes in all cases. The subject has given rise to a great deal of confusion in the minds of teachers and students, chiefly due to a conflict between the theoretical and the practical. It has already been pointed out that, though it may be true perspectively to assume arbitrarily a station-point and a picture-plane, it often results in an unpleasant, because unnatural, effect; it has also been pointed out that certain figures which were described should be sketched comparatively near the center of vision; this was done to avoid the distortions likely to be noticed when the subject is large as compared with the assumed distance of the observer from the picture. This latter is a truism, that applies to a drawing on a flat plane when the subject trespasses outside of the bounds of a very small region, not easily defined, directly in front of the observer, and the distortions may become noticeable and cause the drawing to be criticised as incorrect, from the fact alone that, except by rare chance, any one looking at the result does not occupy the position of the assumed station-point. In some cases the false effect can be obviated by removing the station-point to a greater distance; in other cases so-called errors or compromises are introduced to make the drawing pleasing from any point of view. These last may not really be errors; they may be what would happen if the drawing were made upon the inside of a cylindrical or prismatic surface composed of flat planes, afterwards unrolled with the observer's position assumed to be somewhere on the axis.

It is impossible here to go into a full discussion of perspective adaptations or of distortions; suffice it that the latter are caused in most all cases by the observer's being assumed too near to the picture-plane relative to the size of the picture. Distortions in any pronounced form are not likely to give much trouble except in drawings of great size or including a large field. They are still less likely to occur when the draftsman is rendering a subject which is in front of him; they are apt to give some trouble at first, however, when he is drawing from knowledge alone. The 'apparent incorrectness' of a picture just spoken of can often be anticipated and overcome if the subject be drawn as if on a series of flat planes, the picture-planes being vertical and different parts of the subject drawn on that plane which is perpendicular, at the time, to the direction in which the observer is looking. Unless the drawing take in a very large field composed of scattered large subjects or numerous small ones, this compromise is all that is necessary.

64. Compromises made in the Case of Circles.—The circle is a form which shows most quickly a change in appearance when removed from a position directly in front of the observer. So much does it show this that compromises must always be made in its case when the removal is pronounced or when there are a number of circular forms to deal with. The compromises that are made vary somewhat according to the kind of subject dealt with. Fig. 33 shows one frequently made use of in the case of architectural subjects, a long row of pillars crowned by semicircular arches. The major axes of the ellipses are all

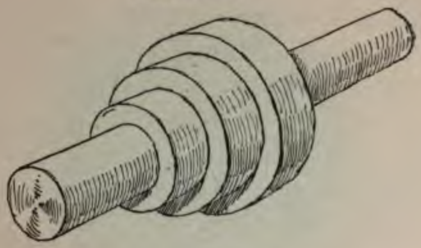
FIG. 33.



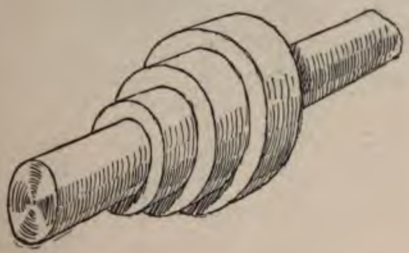
parallel and vertical. In such a case custom seems to require this construction as appearing more correct, while a glance at Fig. 30 will show that in the vertical circle, and in the oblique one, such is not true perspective; the axis of the first would be vertical only if the horizon passed through the center; immediately the circle is elevated above or below it the axis inclines somewhat. This compromise in architectural subjects

FIG. 34

CORRECT.



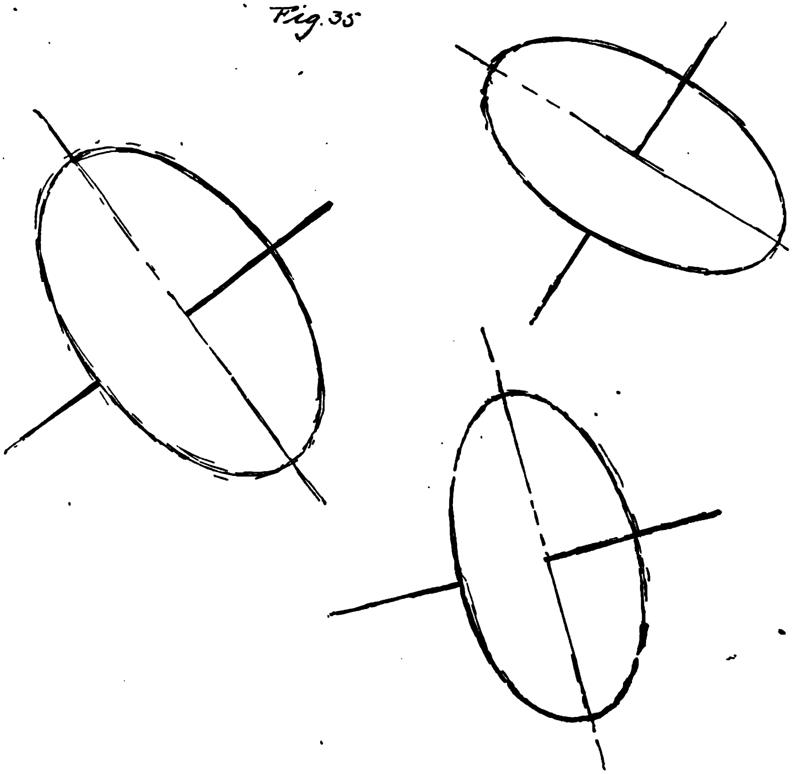
INCORRECT.



is not invariable, though frequently used; the most usual treatment is one made in drawings of small subjects, such as household articles of furniture, machinery, etc.; this is exemplified in Fig. 34. It is a set of speed-pulleys; note in the first that the major axes of the various ellipses are drawn perpendicular to the shaft axis, or, what amounts to the same thing, the minor axes lie directly over, and hence are coincident with, the shaft axis.

It is probable that had this sort of compromise been made with the architectural subject just mentioned, the effect would have been an improvement over the true perspective effect and perhaps under some conditions quite as acceptable as the one shown. However, to apply the change made in the architectural subject to the latter example would have led to a very absurd effect, to be seen in the lower figure. The major axes of the ellipses are made vertical and instead of aiding the impression that they are in vertical planes, they actually appear to slant out of the vertical in a direction opposite to those in the upper figure. Yet this last change is one which can unfortunately often be seen illustrated.

Fig. 35



65. Rule for the Sketching of the Circle under all Conditions.—The change made to the ellipses in Fig. 34 is one of general application. For use in sketching it may be stated in the following rule: *An ellipse which is the picture of a circle should have its minor axis coincident with a line which is the picture of the axis* of the circular plane* (see Fig. 35). The word coincident, as here used, means that the two are coincident in the drawing. The real condition is that a plane passed through the minor axis of the ellipse and the axis of the circular plane also passes through the station-point, hence both are seen as one and the same line.

In practice the circle is very rarely encountered alone, simply as a plane surface but when it does appear, which is frequently, it is apt to be associated with a cylindrical or conical surface. In these cases the axis of the plane spoken of becomes the axis of the cylinder or the cone respectively.

66. The Circular Cylinder and its Interpretations.—The right circular cylinder, or the cylinder which has circles for its bases perpendicular to the axis, is a very common form and deserves some special attention. Its construction under all conditions may be typified in three problems:

(1) To draw a cylinder, given the direction of its axis.

(2) Given the direction of the contour elements of a cylinder, to draw its bases.

(3) Given the drawing of one end, to complete the cylinder.

These problems are illustrated in Fig. 36, at *a*, *b*, and *c*.

(1) A direction may be given for a line of overhead shafting. The axis of the shaft, the elements of the cylinder of the shaft, together with those of the surfaces of any wheels mounted upon it, have a common vanishing-point, hence the ellipses can be drawn by the rule given.

(2) If it were required to turn a given square prism into a cylinder whose surface was tangent to the surfaces of the prism, then the conditions would be those shown at *b*. The

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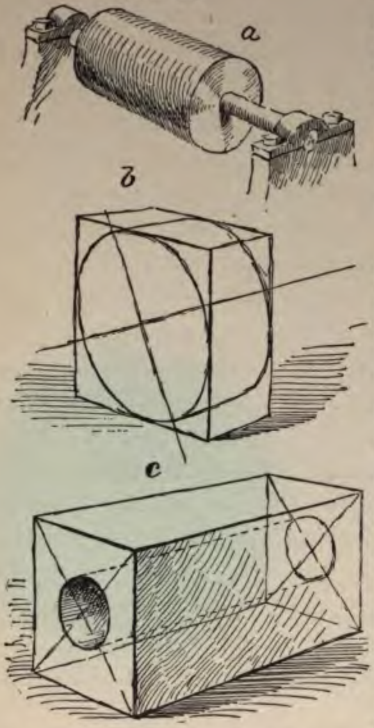
* See section 28.

receding and converging short edges of the prism would give the direction for the contour elements. Since these are all parallel to the axis of the prism and are equidistant from it, the result, by following the rule, is to cause the major axes of the ellipses to make equal angles with the contour elements.

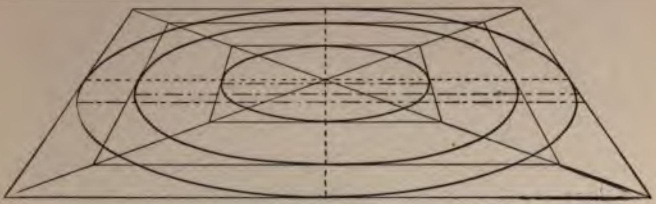
FIG. 36



(3) If a square block were given with a circular opening through it and it were required to show the line of this opening on the opposite face by completing in dotted construction, then *c* would show how this was done. The contour elements are drawn at equal angles to the major axis of the ellipse. Of course if the opening goes through perpendicularly to the square face, the receding edges perpendicular to that face are parallel to the cylindrical elements, and this would also suggest its solution. A few additional problems in the drawing of the circle are subjoined at the end of this chapter.

67. The Perspective of Concentric Circles.—There is one other convenient but slight modification deserving of notice resorted to in drawing the pictures of circles, used in the case of concentric circles. It can be seen in the simple perspective sketch of Fig. 37. Each of the ellipses is circumscribed by the picture of a square, and all of these squares have the same 'picture of the center.' The ellipses, however, do not all have coincident major axes; the distances between any two of the curves are equal at either extremity of the major axes, because the latter are the pictures of chords of the circles which are all parallel to the picture-plane; but along their common minor axis the distances between them vary according to the law

FIG. 37



that measurements and objects become smaller the further distant they are. This last difference, though not always very apparent, is yet present no matter how nearly the circles are of the same size. It depends upon the relation of the sizes of the circles and upon the position of their plane; within limits it is most pronounced when the plane is viewed nearly tangentially, least when viewed nearly perpendicularly. If a subject drawn be complicated and full of concentric circles in parallel planes, and except the difference in size is very great, it is customary, and quite allowable, in general to ignore it. It is not offensive to the eye and simplifies the constructive features very much.

68. Problems in Circles and Circular Curves.—*Problem 1.* Copy the sketch of the square block as in Fig. 38, modifying its form so as to have the shape shown in the diagram to one side, namely, ends rounded off cylindrically and holes shown through the centers of the cylindrical ends, the size of the holes immaterial.

Problem 2. Copy the sketch of the square-ended block as in Fig. 39, and modify its form according to the diagram at one side, namely, cut off the corners in concave cylindrical surfaces and show a hole through the center. The dimensions are in units and on the basis of one edge of the square face of the block being one unit.

Fig. 38.

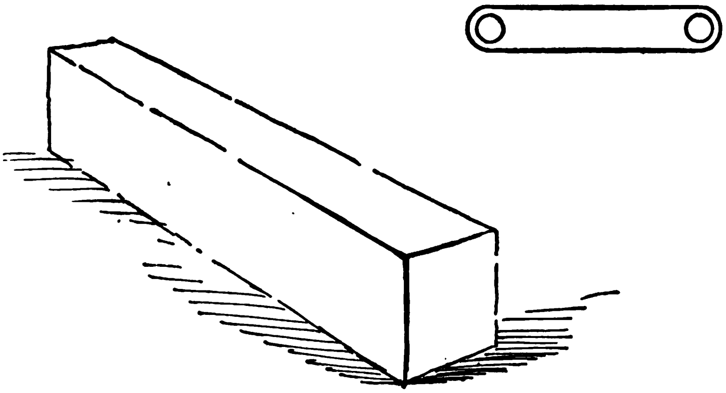
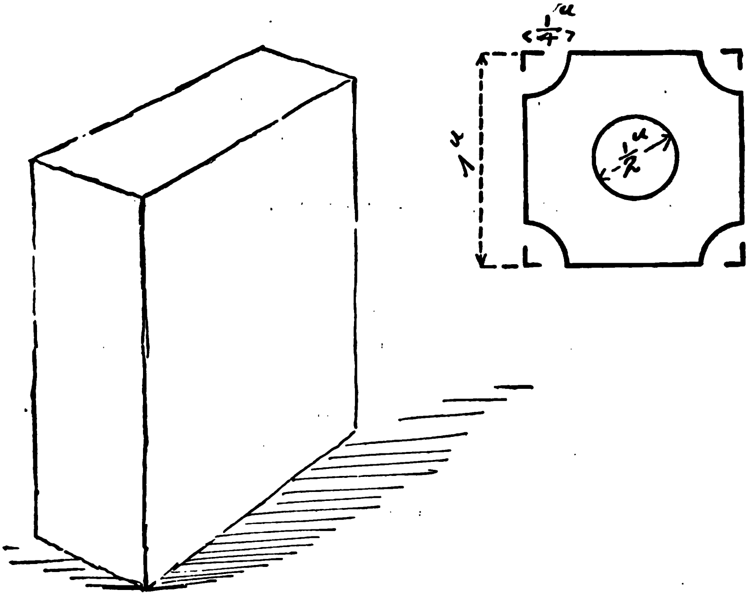


Fig. 39.



Problem 3. Sketch the square block of Fig. 40, modified as shown in the diagram to one side, namely, turn it into a quatrefoil plate, in which the foils are tangent to the edges of the plate.*

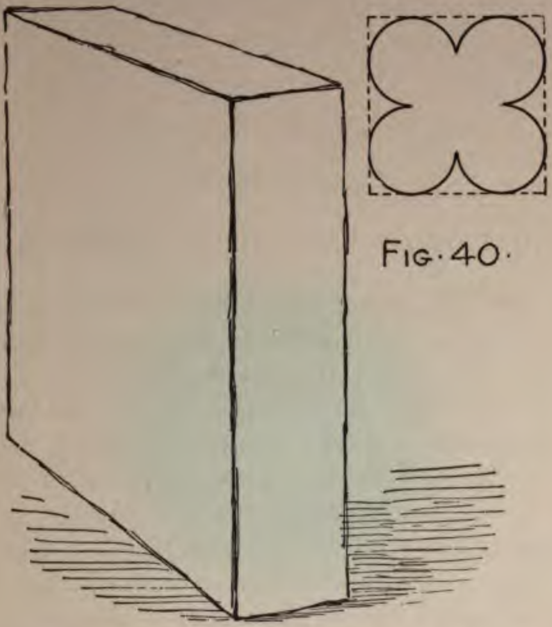


FIG. 40.

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* The figures might be traced and transferred, since they are simply given for the treatment of the curve.

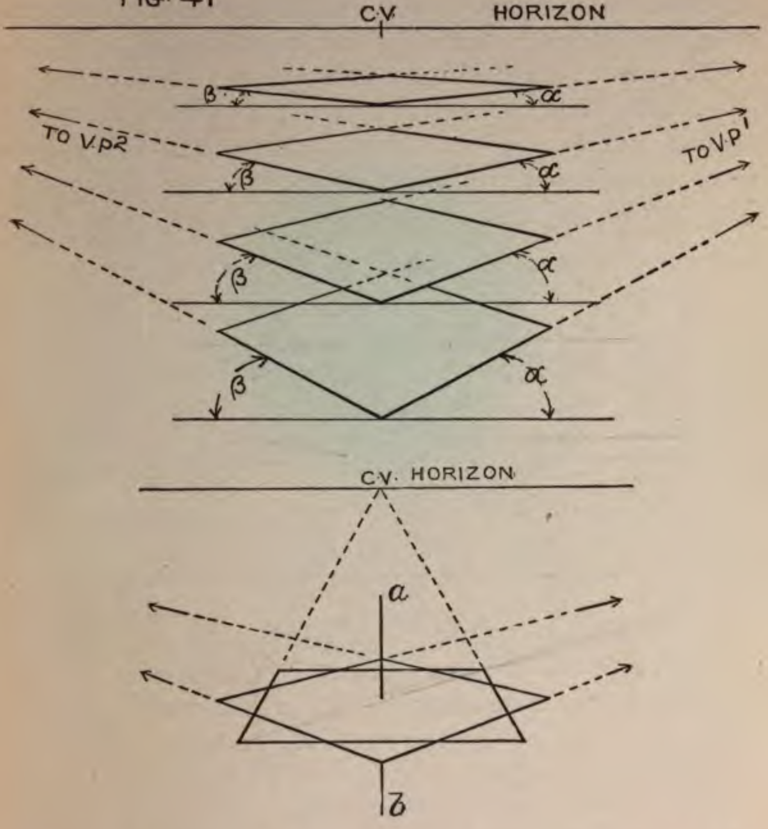
CHAPTER VI.

THE ESTIMATION OF THE VALUES OF PLANE ANGLES.

69. General Statement.—In order that the draftsman may have command of the conditions of a picture without the assumption of a certain station-point and distance of picture-plane, in other words, that he may be able to sketch in at once, approximately correctly, the fundamental lines of a subject with the proper point of view desired, it becomes very necessary to have a clear idea of the values of angles as seen from various points of view. For example, to take an extremely simple case, if sketching a rectangular solid standing on one of its bases, first of all it may be necessary to know what the directions of the receding edges should be relative to each other and the picture-plane in order that a given amount of the top face be seen; secondly, what foreshortening the vertical faces should have to give the proper impression of the form sketched. The second is of course dependent upon the first. Unless the draftsman have some such facility to sketch what he wishes off-hand, he is likely to be very seriously hampered by not being able with perspective rules alone to get the exact conditions of the picture wished, as has been before alluded to. This subject is a very important one.

70. Upon what the Perspective Value of a Plane Angle Depends.—In the first place it must be recognized that the value of any given plane angle between two lines depends upon two conditions: (1) the angle at which the plane is viewed; (2) the direction from which the angle is viewed. This can be seen exemplified in the upper part of Fig. 41. Several square planes are drawn, one above the other, with edges equally inclined to the picture-plane. The angles α and β in each case are obviously the pictures of equal angles, yet they are not all drawn equal. The angle at which each plane is

FIG. 41

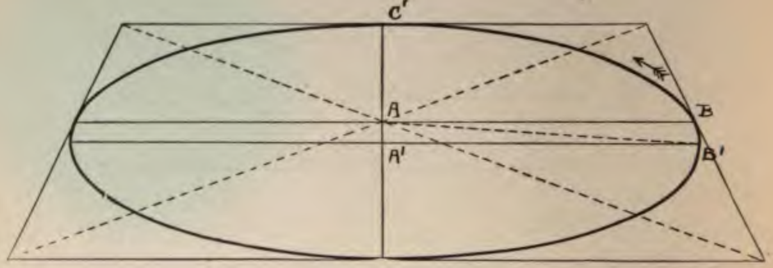


viewed is different from that of the others. Again, the four angles of any one of the square planes are known to be equal, yet they are not so in the picture because the eye is viewing each of the four differently. This last fact appears clearer by reference to the lower part of the figure, where one of the square planes is shown in a couple of different positions, as if it had been rotated into them about its axis *ab*; the four angles of the square are seen to have different values in the different positions. The direction from which the angle is viewed governs its appearance.

71. The Directed Plane Formed by a Moving Line.—The conditions which govern the value of an angle viewed from different positions, as in these figures, may be derived as follows:

If a line of definite length is rotated in a plane about one of its ends as a center, the other end will generate or describe a circle, and the whole line a circular plane, which for convenience in reference may be called a directed circular plane. Fig. 42 shows a horizontal plane of this character, directly in front of the observer, the initial position of the line being shown at *AB*, and the motion of it, in generating the plane, assumed to be counter-clockwise. The semi-major axis of the ellipse bounding the plane is at *A′B′*. It has been seen that these two lines would not exactly coincide, but it has also been suggested that in general the difference may be neglected, so that the statement can at once be made, without appreciable error, that *A′B′* is the greatest apparent length of the moving line, and that *A′C′* or *A′C* is the least.* In going through one complete rotation, then, the line will pass in each quadrant

Fig 42



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* The real position of the semi-diameter of the circle when having its greatest apparent length is somewhere between *AB* and *AB′*.

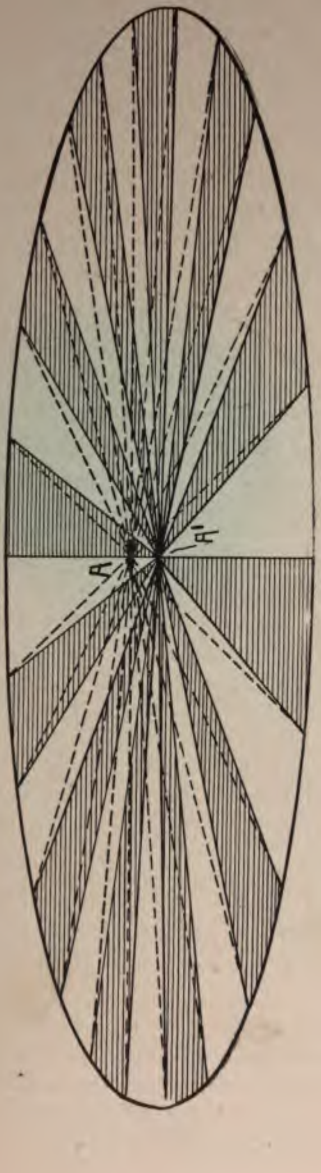

Fig. 43

through all values between $A'B'$ and $A'C$. It can also be seen that, as the line moves from $A'B'$ to $A'C'$, its extremity on the curve passes along a portion of the latter which grows less and less foreshortened, so that if the line were to move intermittently into equidistant positions, the equal angles formed between any two of its positions will subtend unequal arcs, in reality arcs, starting from B', which are decreasingly foreshortened. Fig. 43 is introduced to make this last fact more apparent. It shows a division of the plane into angles of 15°, the alternate ones being shaded to make the relations between them more prominent. If the picture of the center of the circle were taken as the center of the rotating line, the angles would appear as shown by the dotted lines, and it can easily be seen that no appreciable error in effect is involved in transferring the center of rotation to the center of the ellipse.*

72. A Test for the Estimation of Angles.—The preceding paragraph gives a key for the approximate estimation of the value of any plane angle. The conditions which may be assumed in order to effect this can be expressed as follows: *A small angle formed between two equal and limited lines will appear larger or smaller according to whether the angle is subtended by an arc which lies in the neighborhood of the minor or major axis of the directed plane of rotation of the lines.*

A little practice in the sketching of problems containing specified angles, a few of which are to be found at the end of this chapter, will make clear the application of this test. It must be understood that its use is to enable one, in sketching, to readily estimate an angle without rigid perspective processes. The words 'small angle' used above have a definite meaning. As an angle is taken larger and larger it will be subtended by a larger arc, and if this arc cover a whole quadrant or more of the directed circular plane, then it is not so easy to estimate its foreshortening, though it can be done by noting which axis lies within the angle or in the neighborhood of which axis the major part of the angle lies.

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* If it is desired to reproduce this sketch carefully by perspective, it can easily be done. Draw a circle and circumscribe it by a square. Prolong a number of radii of the circle until they cut the sides of the square. Put the circle and square into perspective by transferring the points by the method of parallels and diagonals.

It would be well for the student to sketch out one or two of the directed planes similar to Fig. 43, using larger angles, and note the differences in values.

73. Limiting Values of any Angle.—In section 69 it was remarked that, to show a certain effect, the problem may arise of how to properly represent a simple figure like a rectangular solid. It often happens in laying out a sketch that this is one of the important things required to be done; for example, to represent the direction of the two horizontal edges of a rectangular solid placed so that it shows two vertical faces; it may be in the shape of a building whose two walls in view are oblique to the picture-plane, or it may be in the bed of a piece of machinery. What apparent value should the two edges which are at right angles to each other have in the drawing? The greatest value the right angle may have is 180°, when the plane of the angle passes through the observer's eye, and when the sides of the angle recede from or approach the observer on opposite sides of the station-point, i.e., to the right and left. The least value is zero degrees, when the plane of the angle is in a similar position, but the sides of the angle recede from or approach the observer on the same side of the station-point. The actual value of 90° means that the plane of the angle is parallel to the P.P. If from the first, or 180°, value the plane is so moved that it is viewed at an increasing angle, the apparent value decreases; if moved parallel to itself, there will be some limiting value, not easily determined, beyond which distortion would begin to be objectionable. An ordinary drawing does not represent objects as if very far from a line of direction from the eye to the P.P. The limit mentioned is not a fixed one; it is certain, however, that it would be within the actual value of the angle, because this actual value could only occur when the plane of the angle is parallel to the picture-plane. To make a statement which is perfectly general it may be said that '*the less the obliquity of the plane of an angle to the picture-plane, the nearer to the actual value of the angle will it be possible, from considerations of position, for the limiting value of the angle to appear in a drawing.*'

This matter can be best investigated by taking a concrete case, and by drawing in linear perspective some simple subject, say a rectangular plane under several different conditions as to position, note how far from the center of vision it is possible to go before distortion is apparent, and under such conditions what the value of the angle is.

74. Comparing Unknown with Known Angles and the Sketch Process in General.—The average draftsman is not as familiar with the values of angles such as 10°, 15°, 32°, etc., as he is with those of 30°, 45°, 60°, and 90°; but the former can be compared with the latter in any case, just as any form in a picture which is uncertain can be compared with any other which is correctly drawn.

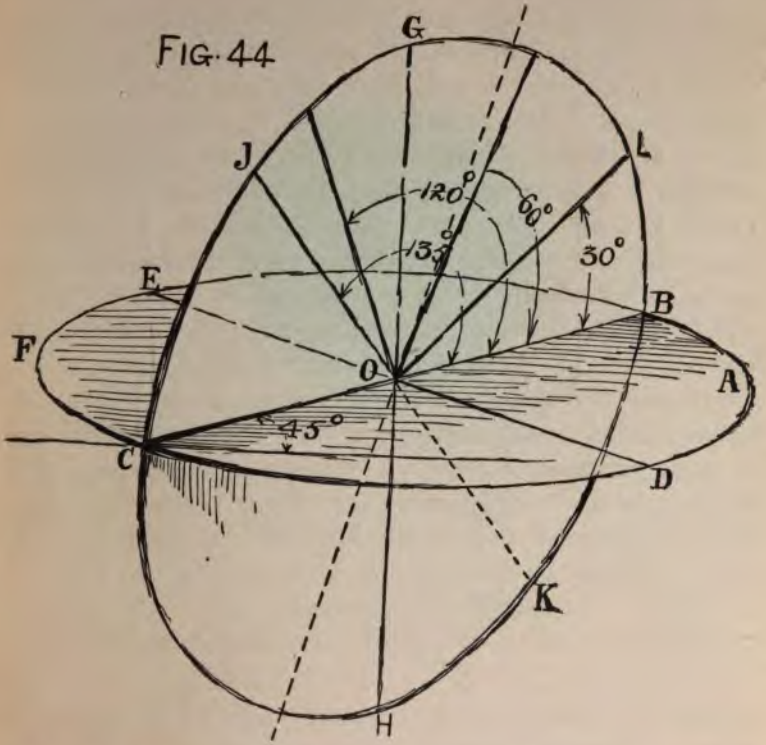
The general process of sketching any plane angle would be, first, to ascertain at what degree of obliquity the plane of the angle is viewed. This can generally be effected by sketching the directed circular plane formed by the rotation of a side of the angle, using the vertex as center, indicating its major and minor axes; then the situation of the angle with respect to either axis will determine approximately its size.

75. Illustrative Examples of Circular-directed Planes.—It now only remains to apply the general directions given to some specific cases by a few examples worked out in detail..

Problem 1. Sketch the directed circular planes of two equal and limited lines touching each other at their extremities, one of which rotates horizontally about their common point, the other rotates in a vertical plane about it, the latter plane to stand at an angle of 45° to the P.P. In this latter plane also sketch the positions of the line when it has rotated through angles of 30°, 45°, 60°, 120°, and 135°, respectively, from any convenient position of reference.

Fig. 44 shows one solution.* The plane of the first rotating line, namely, *DBEC*, is sketched at random, due observance being had for the fact that if horizontal the minor axis of the ellipse could not be very great in an ordinary picture of the circle. The second directed plane being vertical, its angle

FIG. 44



with the P.P. is measured by means of two horizontal lines one of which is in its own plane, the other is either in or parallel to the P.P. *OB* is the first line; *OA*, the position of the major axis of the ellipse of the directed plane, can be taken as the second or, what is the same thing, one parallel to it through *C*, these two lines making with each other an angle of 45°, estimated by the test given in section 72 (note that it will be considerably less than one-half of the 90° between the major and minor axes). The two moving lines being equal, *B* and *C* will be two points in the second directed plane. When *OB* has rotated in this latter plane through an angle of 90° it would reach a position coincident with either *OG* or *OH* according to the direction of rotation assumed.* *OG* and *OH* are approximately equal in length to *OA*. Now, by the rule given in section 65, the second plane will have its minor axis coincident with the axis of its plane; this latter axis must be horizonal and make the complement of the angle with the P.P. that the horizontal line *CB* does. *EOD* is the axis of the plane, the angle *EOF* being actually equal in this case to the angle *BOA*. Given the direction of the axes, and the four points in the curve, it can be drawn approximately. The difference in length between *OG* and *OA* is zero when the plane *BGCH* is either parallel or perpendicular to the P.P., and is a maximum at some point, not necessary to determine, in between. The level of the eye also governs the difference; it is zero when the level is identical with the horizontal directed plane, and increases as the level is raised or lowered.

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* A variety of solutions might be made depending upon the level of the horizontal directed plane; also the angle with the P.P. of the vertical directed plane could be oppositely directed.

Problem 2. Sketch the directed circular planes of two equal lines touching each other at their extremities, one of which rotates obliquely about their common point in a plane which makes an angle of 30° to the horizontal plane, and in which a horizontal diameter would stand at 30° to the P.P.; the second line rotates about the same point, also obliquely, in a plane which makes an angle of 45° to the horizontal plane

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* *OG* and *OH* are actually slightly less than *OA*, because *O* is assumed as the center of rotation, which it is not in reality; in general the difference is slight, but it must be sufficient, in such a problem as that above, that were a circle described with *O* as its center and *OA* as its radius, it would be tangent to both directed planes. In problems in which one process is a check upon another it may be necessary to reconcile small discrepancies that arise from the incorrect assumption made. The major axes of these directed planes should be taken the same length.

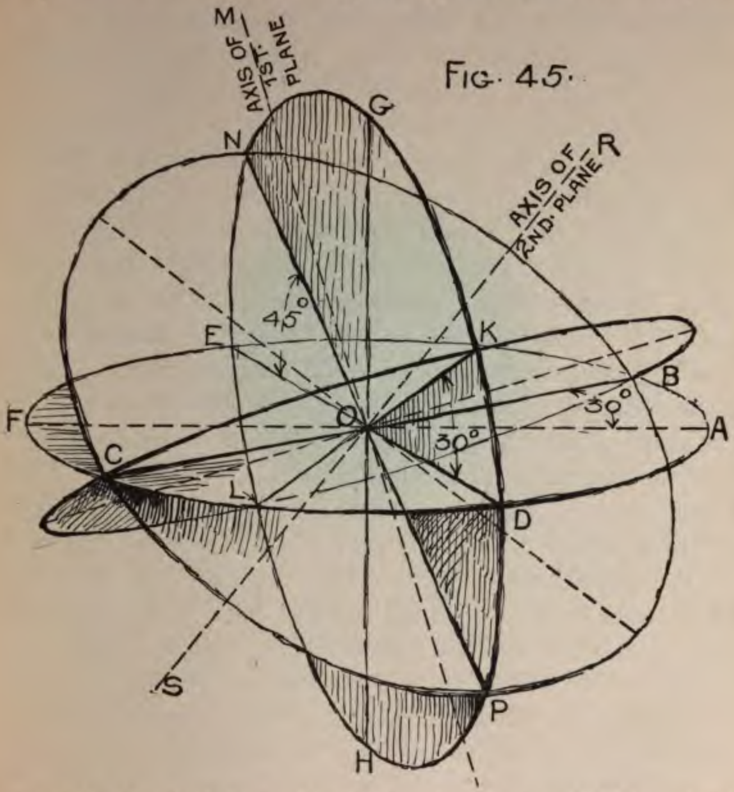


FIG. 45.

oppositely directed from the first and a horizontal diameter in which coincides with that of the first (see Fig. 45).

The line of intersection of these two planes will be the horizontal diameter spoken of above and may be for convenience also regarded as a horizontal trace between the planes and an auxiliary horizontal plane which could be passed through it. The solution of the angles can be further facilitated by considering this auxiliary plane as formed by the rotation horizontally of a line equal in length to the two given lines and about the same center of rotation. *ABFC* is this plane similar to that given in Problem 1. Now by means of this auxiliary horizontal reference plane it will be possible to estimate the obliquity of the required planes with respect to any horizontal plane, and that of the line of intersection of the given planes with the P.P.; also, by using the same length of rotating line for the auxiliary plane, points can be at once obtained in the required planes, for *COB* is the horizontal line of intersection of the required planes estimated as at 30° to the P.P., and since the rotating lines are equal in all cases *C* and *B* are points in the two required planes. To measure the angle of the first required plane with *ABFC* a line must be taken in each of these planes perpendicular to their intersection, *COB*. The lines can be best obtained by passing a second auxiliary plane perpendicular to *COB* and making it a directed plane as before with *O* as a center and formed by a rotating line equal to the others. *DGEH* will be this plane constructed after the manner of the vertical planes in Problem 1. *EOD* is a line cut out of the horizontal plane, and *OK*, estimated at 30° to it, will be a line cut from the first required plane, and *K*, the point where this line cuts the curve of the directed plane, must be a point in the curve of the first required plane; *L*, a point symmetrically opposite to it, will be another point. We have now four points in the curve, and it remains to locate the axes of the ellipse representing this plane; these depend upon the axis of the plane. This last must be a line perpendicular perspectively to *LK* and *CB*, namely, a line in the plane *DGEH* making the angle of 90° with *OK* or 60° with *OE*; *OM* will be this line. The minor axis of the ellipse is superimposed on *OM*, and the ellipse representing the directed plane can be readily drawn.

The second required plane is located in a similar manner. *ON* is drawn at 45° to *OE*, oppositely directed from the angle *KOD*, and *N* is a point in the second required plane; *P*, opposite to *N* on *ON* prolonged, will be another point; *C* and

B will also be two points. The axis of this circular plane will be *RS*, a line in the plane *DGEH* making 90° with *ON* or 45° with *DO*. The minor axis of the ellipse of the required plane will coincide with *RS*, and the curve can be readily drawn.

76. Estimation of the Foreshortening of a Plane Surface.—The foreshortening of any limited plane surface depends entirely upon its obliquity to the P.P., provided it is very close to the C.V. If this angle of obliquity is known, it is possible to make a fairly close estimate of the amount of foreshortening of that dimension of the surface, representing its angle with the P.P. as follows: Let *AB*, Fig. 46, stand for a certain dimension of a surface when it is parallel to the P.P. When turned through the angle α this dimension will have the position *A′B*. If the observer were to be infinitely distant from the P.P., then *DB* would be the projected dimension of *A′B* upon the P.P. Since the distance of the observer is finite, *Bb′* and *A′d′* might be the converging lines to the station-point, hence the picture of *A′B* would be actually slightly less than *DB*, for example, *dB*. However, as the observer in most pictures should be assumed at quite a distance, the difference will be almost small enough in many cases to be negligible, and the projected length taken as the perspective dimension.

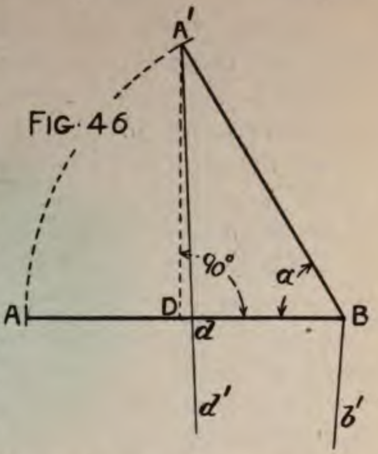


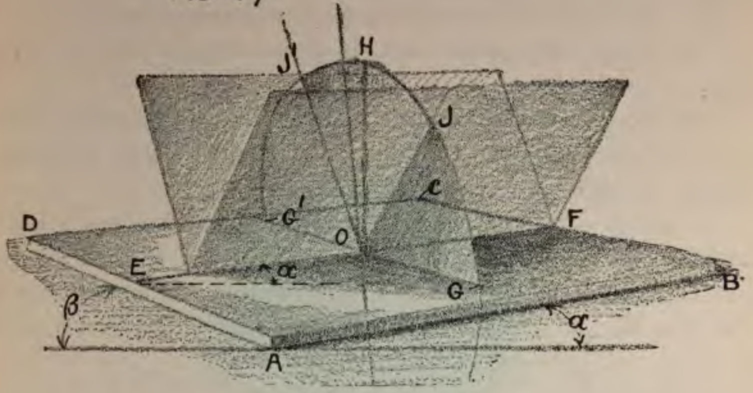
FIG. 46

77. Observations upon the Application of the Directed Plane to the Measurement of Angles.—If it is desired to sketch two intersecting lines having a definite angle between them, the plane of the two lines must first be located This can be effected by means of some line or lines which define its position; if one of the lines first mentioned or some fraction thereof is pictured as rotated about the intersection of the two within the plane of the angle, this position can be found by means of the plane generated, and the value of the angle can then be estimated. The direction of the axis of the directed plane of the angle can be given by means of a second directed plane perpendicular to the first; the axis being perspectively at 90° to the line of intersection of the two in the plane of the second. This would be the general method.

In dealing with plane surfaces and estimating their angles with each other, the fact must be kept clearly in mind that if a plane is passed perpendicular to two intersecting planes, i.e., perpendicular to their intersection, it will cut a line from each plane, the plane angle between which will be a measure of the angle between the planes; this third or auxiliary plane can be profitably used in any such calculation in sketching, and it can be made to assume the shape of a directed plane formed by the rotation of a line which continues in all positions to be perpendicular to the line of intersection of the two planes whose angle with each other is to be determined.

To illustrate this, let *ABCD* in Fig. 47 be any plane surface taken for convenience as a rectangular plane, placed horizontally; the angle α which *AB* makes with the P.P. is the complement of the angle β which the edge *AD* makes with it. Let it be required to place another plane surface intersecting this one and at an angle of 60° to it. The line of intersection of this plane with *ABCD* can be assumed anywhere in the latter and hence at any angle with the P.P. For simplicity take it as *EF*, a line parallel to *AB*; its angle with the P.P. will be α. In accordance with what was said above, it will be necessary to sketch an auxiliary directed plane perpendicular to *ABCD* and to the 60° plane yet to be drawn, cutting out of each lines which are at 60° to each other; to do this *OG* may at once be sketched parallel to *AD* as the line assumed to be cut by this auxiliary plane from *ABCD*, and *O* taken as the centre of the plane. The dimensional value perspectively of this plane cannot be represented correctly until some numerical value is given to β, the angle which *OG* makes with the P.P. When this is done the dimensional value or relative foreshortening of the plane can be represented by drawing *OH* of a length nearly equal to what *OG* would be when parallel to the P.P., i.e., slightly less than it, since the directed plane is below the eye (see section 75, problem 1). *OJ* or *OJ′* may be sketched at 60° to *G′OG*, either one as a line cut out in the required plane by the auxiliary plane, so that a plane passed through *EOF* and either *OJ* or *OJ′* will be the oblique plane desired. Suppose the required plane is, like *ABCD*, a rectangular plane, then lines intersecting each other on *EOF* and drawn perspectively parallel to *OJ* or *OJ′* will represent the edges of the plane.

FIG. 47



The above shows a general plan of procedure for sketching two or more planes at definite angles to each other. As an aid to the working of problems in sketching where the estimation of angles is involved, the several facts and stages that should be observed are summarized below:

(*a*) The angle between two given intersecting planes is measured by means of the plane angle between two intersecting lines perpendicular to the line of intersection of the given planes.

(*b*) The plane of this angle may be considered as a portion of a directed plane formed by the rotation of a line within the plane of the angle.

(*c*) The axis of the directed plane is then the line of intersection of the two given planes.

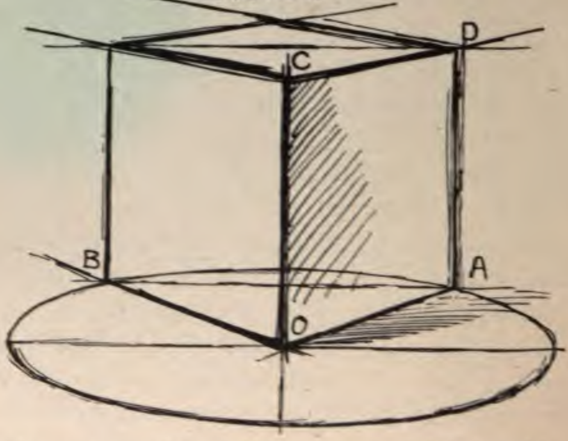
(*d*) The minor axis of the ellipse which is the picture of this directed plane is coincident with this line of intersection.

(*e*) Finally the major axis is perpendicular to the minor axis.

Problem 3. Sketch a cube standing on a horizontal plane, two vertical faces equally inclined to the P.P., and a top face below the level of the eye.

Solution.—Since the exact level is not fixed, there is nothing to limit the magnitude of the base or the top, so that, in sketching the directed plane for estimating the inclination of the receding edges, the ratio of the minor to the major axes can be given any value. However, in all sketching, unless there is something definite specified, it will be well not to put the subject very far from the C.V., i.e., the ratio between the axes should be small. Fig. 48 shows the construction and the result. The plane of the base is considered first for greater convenience and an ellipse sketched, then the inclination of the

FIG. 48.



edges of the base estimated approximately, and the perimeter of the ellipse assumed to cut off the edges the proper length. *AB* will be a diagonal of the base, horizontal in the sketch; *OC* is taken slightly less than the length of the semi-major axis of the ellipse. The receding edges of the top are then sketched so as to seem to converge towards those of the base, one diagonal being horizontal.

Problem 4. Sketch the same cube as if rotated on its rear right-hand edge so that the face which was in contact with the horizontal plane now stands at an angle of 30° to it.

Here the directed plane of rotation most convenient to start from is that of the face *COA*, with *A*, one corner of the base, as a center. Fig. 49 shows the conditions when finished. The line *OA* moves in the plane *OAC* to *O'A* through 30° measured from the line *OA*. The ellipse representing the directed plane of rotation of *OA* in this case cannot be located as readily, but the small portion of the curve which is needed can be approxi-

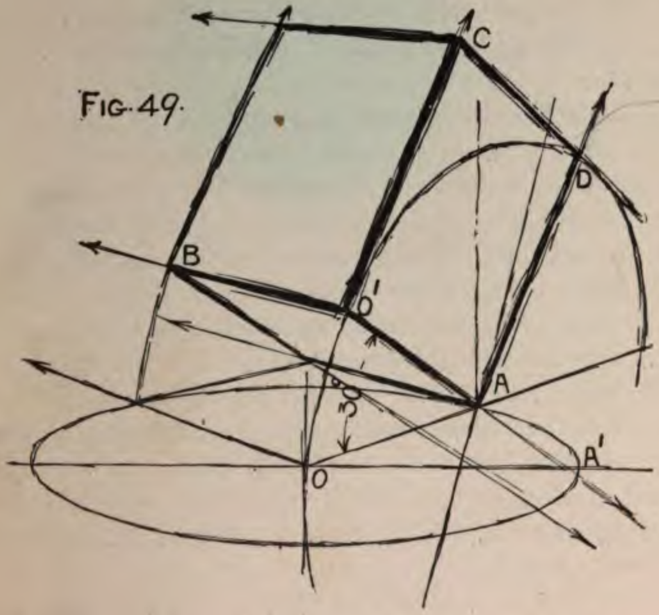


FIG. 49.

mated through *O* and points vertically above and below *A* distant from it a length equal to *OA'*, when unforeshortened, near enough for all practical purposes. *AD* will be at an angle of 30° with its former vertical position. The arrows show the direction of the vanishing-points; of course *O'B* and the corresponding parallel edges of the cube move parallel to the axis of rotation and still have the same vanishing-point. The degree of convergence of the several systems of lines can be approximated closely enough by comparing the relative inclinations of the several sets of edges with respect to the P.P.; the system of which *AD* is one is least inclined, and that of *OA* the most. In estimating this relative inclination it must be borne in mind that the angle of the lines with the P.P., in each case, is measured by means of a plane passed through an edge and perpendicular to the P.P. It is a simple matter in the case of the edge *O'B*, but not so in the case of the lines *AD* or *CD*.

78. Additional Problems for Practice.—Of course it would not be desirable, in sketching practically, to construct a directed plane whenever it is necessary to estimate an angle; after some practice and with a knowledge of the law which governs the values, the ellipse forming the directed plane can be simply a mental picture to guide in drawing. Below are given a few additional problems involving the estimation of angles together with the drawing of circular curves.

Problem 5. Sketch the directed planes formed by the rotation of three equal lines, their extremities meeting in a common point; one line rotates in a horizontal plane, the other two in vertical planes, each at an angle of 30° to the P.P. but oppositely directed, all of the planes to be slightly below the level of the eye.

Problem 6. Sketch a square plane standing on the horizontal plane on one of its edges, the edge to be at 45° to the P.P. and the plane of the square at 60° to the horizontal plane.

Problem 7. Sketch the following: First, a square plane standing on the horizontal plane on one of its edges, the edge to be at 60° to the P.P., and the plane of the square at 45° to the horizontal plane. Second, sketch another square plane similarly situated but its angle with the horizontal plane oppositely directed, and in addition cutting the first square in the center so that their diameters are coincident and the inclined edges of both squares lying in two vertical planes.

Problem 8. Sketch a square plane making an angle of about 20° with the horizontal plane, the angle with the P.P. of its horizontal edge may be optional; within this inscribe a square, its corners touching the middle of the sides of the first; then turn the last square into a block approximately one-quarter of the edge of the square in thickness.

Problem 9. Sketch a cube resting on a corner, the face and edge which meet at the corner and which are perpendicular to each other making equal angles with the horizontal plane.

Note.—It would perhaps be the most instructive plan to first sketch a cube after the manner of problem 1, then move it to the position of problem 3, using very faint sketch-lines for the first which could afterwards be erased.

Problem 10. Sketch a right circular cylinder, one unit in diameter and two units long, lying on one of its elements on a horizontal plane, the circular base at 20° to the P.P.

Note.—Circumscribe the cylinder by a square prism; this, it will be noticed, has the proportions of two cubes placed side by side; use the short edge of the prism as the line generating the directed plane.

Problem 11. Sketch a square disk six units long on its square edge, and one unit thick, resting on a long edge, the plane of the disk at 30° to the horizontal plane. Support it by a thin vertical rod touching the middle of the raised edge. Represent a circular hole through the disk, four units in diameter and concentric with it.

CHAPTER VII.

METHODS OF APPLICATION AND DISTORTION.

79. A General Statement.—In section 35 brief mention was made of the application of perspective to sketching, and in section 63 a short discussion was given of the subject of distortions. As it is very desirable to know more fully in just what measure perspective principles can be employed in sketching, and what limits are set to the interpretation by them of objects not present before the observer, it is proposed in this chapter to make a fuller inquiry into these two subjects.

80. Perspective Modified to Suit Modern Practice.—As was stated in the first section, perspective is both a science and an art. It is a science in its narrower meaning and an art in its broader one. All of that which a painter does to make a picture look real, the effects obtained, each of which the critic may call by a different name, may, after all, come under the one general head of perspective, but not, however, as it is commonly understood. The artist's perspective is usually not based upon an intimate knowledge of the science; on the contrary it is much more often unscientific and for that reason draws severe criticism from those who would wish to see the painter lay out his work on a drafting-table. There is an antagonism between the two arising from this that renders it difficult to establish the true status of perspective principles. It is true that, though the scientific or linear perspective is educative and capable of an infinite variety of problems in applied geometry, yet its uses to the artist are to-day fewer than they have been in the past. Photography is chiefly responsible for this, as it has made it possible for the draftsman to dispense with cumbersome perspective processes when dealing with complicated subjects. On the other hand it would seem as though the science is destined to have a new life in a different way. The ability to draw readily is becoming more and more an accomplishment of the people at large, to speak through the medium of a drawing as by written language, and it is becoming in this respect of considerable importance in the crafts. It is impossible for every one to become an artist; to draw intuitively, one must draw, if at all, through the medium of a well-defined basis of principles.

In general it may be said that the best effect in a drawing is obtained when the draftsman is least hampered by perspective rules, when he can record impressions intuitively and correctly, the rules being applied unconsciously as far as possible. If the student will try to use the principles here enunciated with that end in view, then no antagonism to them need arise; they should not hinder his work by making it mechanical in its processes, but help by enabling him to see and record the desired result with greater facility. It is true that the tendency of what has been discussed in the previous sections is to empiricize drawing, but it is inevitable that it should do so to a certain extent where natural ability and intuition are wanting, just as any department of human knowledge must be made more or less empirical in order to be available for all.

81. Application of Perspective to Sketching Groups of Geometrical Solids.—The modern plan of beginning the study of drawing is to start immediately with object-drawing, and the objects used are frequently the well-known geometrical solids; these are indeed excellent for the purpose, because with their sharp edges they show clearly their perspective features, and errors in drawing are more apparent and more demonstrable. These perspective features are less and less apparent as forms are used which are rounded or irregular. It is not certain how much assistance in the nature of perspective tests is desirable in the drawing of the geometrical solids; there is always present the danger of depending too much upon perspective and too little upon observation. The solution must rest with the individual; the tests can be used while they assist, but must be set aside if dependence upon them increases to the extent of hindering free observation. Still-life drawing can be practiced of course without any knowledge of perspective, by using the tests of relative sizes and positions of the various parts.* But when it is used perhaps the following outline may be of service.

In the first place it is the general opinion of most teachers that groups should not be drawn by *means* of perspective, but that whatever principles are applied should be in the nature of tests to prove the results of the unaided observation. If this is done, then the tests may be:

(1) Note the harmony which exists in those edges which are parallel; see that in the drawing they seem to converge to one common point; compare the intermediate edges with the extreme edges of the system.

(2) See that objects of equal size but at different distances *do* have in the drawing their proper difference in size.

(3) Compare the positions of various plane surfaces relative to each other and to the imaginary picture-plane.

(4) Estimate the values which various angles should have relative to their size, position, and to each other.

Other minor tests might be enumerated, but the above serve to show the principal ones and the way in which perspective can be applied helpfully. If the drawing does not stand the tests, then, by a careful observation of the *whole*, the difficulties can be found out and the remedies applied.

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*It may be said for the benefit of those to whom the tests alluded to are not familiar that they are: (1) The plumb-line, any piece of thread with a weight at one end, to show what edges are vertical, and also what parts of the subject are directly over other parts; (2) The thread held taut before the eye so that it appears to cover an edge of a solid and thus give its direction; (3) The pencil or stick held at arm's length and measured upon by the hand so that one apparent size can be taken and, without changing the distance, from the eye, of the pencil, transferred to other parts, thus giving the relative sizes together with other tests of a similar character.

As a final direction, definite vanishing-points should never be used in model-drawing.

82. Another Instructive Method of Application.—The time will come, however, when the student will be called upon in practice to sketch without a subject, to draw from imagination, from description, or imperfect data, and it is with this in view that some further work may be done, with profit, that is not directly drawing from the model.

If the position be changed with respect to a group of geometrical solids, so that it is viewed from a different direction, considerable modification of the form can be observed; this can be calculated beforehand very closely by the application of perspective principles, and it is a very instructive exercise to sit in front of a group and at the same time draw it as if it were viewed from another direction at a given angle to the first.

Again, similar modifications can be noticed when the level of the eye is changed with respect to a subject; hence it is likewise instructive to make a drawing of a group, assuming this condition as changed.

Of course it would be very difficult, and involve a great deal of explanation, to show how these exercises may be made in any specific case. But several general hints may be given. In the first place, perspective should not be here used rigidly any more than in the drawing of objects as they are seen. Some chief feature or part of the subject may be lightly sketched with its perspective modified to suit the conditions, and the other parts drawn relative to this one. It is next to impossible to block out such a drawing in a comprehensive manner with equal treatment to the whole until all is finished, as is generally taught in object-drawing;* a step-by-step process is necessary,

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* This statement is made of course as applying to students of limited experience. The artist always blocks out his work and deals in masses related to the whole.

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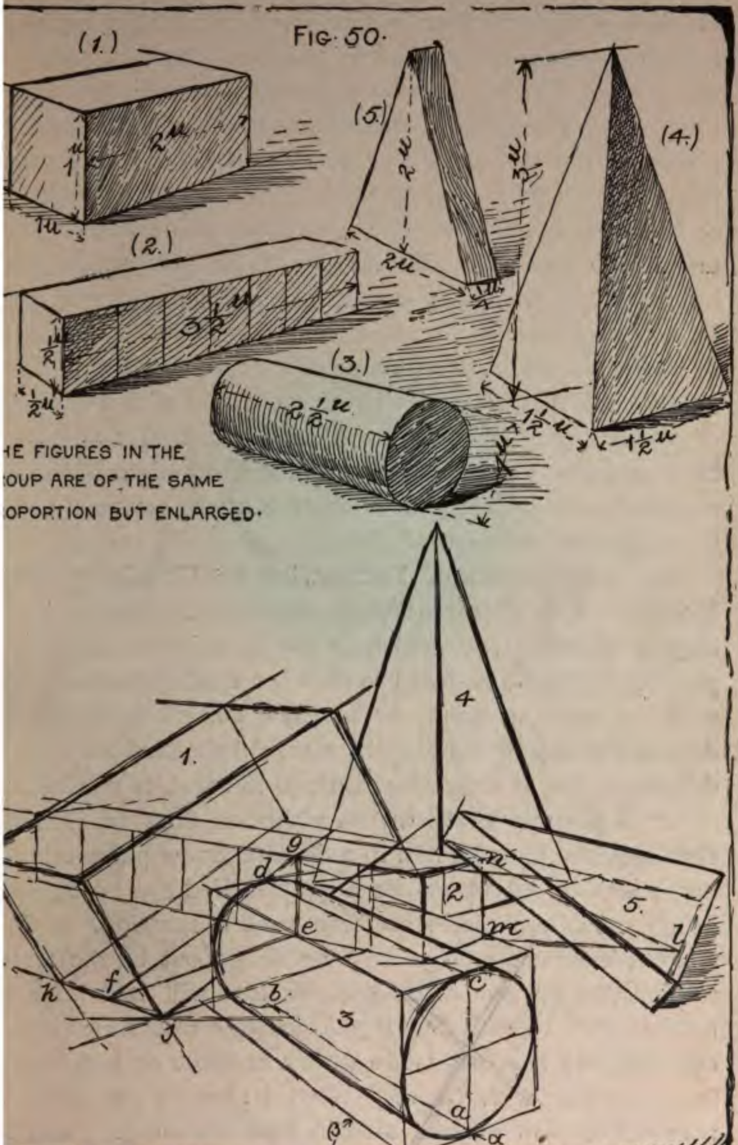
fixing certain easily discerned conditions and making others to harmonize with these. A verification of the assumed conditions may finally be made by taking the position before the subject which would give the result sought, and noting wherein and why the drawing differs from what it should be.

There is another very instructive exercise which is more difficult than either of the two just mentioned, involving as it does greater dependence upon perspective principles and upon experience: it is to select a few of the solids used as models and design a group composed of them. It can be made more or less difficult according to the positions of the various elements of the group. The preliminary work may take the form of a plan and elevation of the intended group sketched very roughly and briefly, so that the perspective features can be analyzed for use in the final drawing, or the group may be drawn directly, choosing that part which seems to promise the least perspective difficulties, and making the rest to harmonize with this. The use of a plan and elevation, just mentioned, has peculiar advantages, because in the crafts it very often becomes desirable to construct a sketch from a working drawing. This will be given fuller discussion later.

83. Illustrative Example.—Fig. 50 is introduced as an example of the last-mentioned exercise, and will serve, in a measure, to indicate how to proceed with the other two, though in them rather less construction may be necessary.

Pictures of the solids used with their proportions in units marked on them are shown at 1, 2, 3, 4, and 5. The plan used in the construction of the group drawing was as follows: Fig. 2 was placed somewhat at random, and the perspective of its long edges determined by sketching eight squares roughly as shown. Fig. 3, first inscribed in a rectangular prism, was next drawn lying on the horizontal plane near number 2, and its axis differently inclined to the picture-plane from that of 2, as shown by the angles sketched on the horizontal plane; these angles could have appropriately been definitely specified, a plan view of the group showing them clearly. The lowest

FIG. 50.



HE FIGURES IN THE
OUP ARE OF THE SAME
OPORTION BUT ENLARGED.

element of the cylinder is seen to be *ab*, and the highest *cd*. It was then desired to incline number 1 against number 2; first a vertical triangular plane *feg* was sketched, the hypothenuse of it, *fg*, was then made the middle line or diameter of a face of the prism number 1 which touches the horizontal plane and which is tangent to the prism number 2. In order to have the edge *jk* on the horizontal plane and the prism tangent to number 2, *jk* must be parallel perspectively to the long edges of 2. Fig. 5 can then be inclined against the prism by a similar process, that is, a triangular plane *lmn* is sketched to determine the inclination of the triangular disk. Lastly, number 4 is placed on its base at the rear of the group with edges equally inclined to the picture-plane. Its altitude can be measured by comparing it with the short edges of the prism number 2, using, if desired, measurements by means of parallels in any convenient vertical plane.

84. Application of Perspective to Sketching without Models.—This deserves some attention because it involves slightly different treatment from the cases just mentioned. In practice it may be required to sketch a modification or addition to some existing form, or to draw something from verbal description, or, again, to make a careful sketch from a working drawing. Let it suffice for all three to consider this last one.

It is possible that such an elaborate drawing is required that recourse must be had to a definite linear perspective layout, which might occur, for example, if the drawing was to be an exhibition piece, or for a neat illustration, or finally, what is most common, a building drawn out by the architect to show in advance what the structure will look like when completed. In such drawings as this the chief forms are fixed carefully and the details are drawn in more or less free-hand in conformity with them. What is known generally as a perspective plan may be used to best advantage, particularly in the architect's drawing; this is described below.

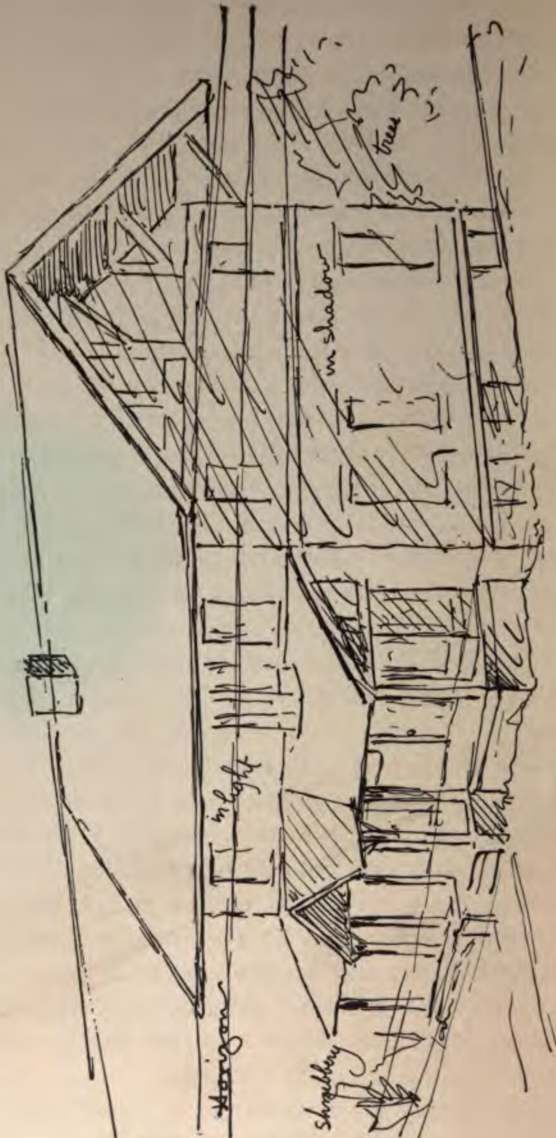
A rough sketch is made at first, free-hand, to determine the general scheme, and from this the vanishing-points of several of the principal systems of lines established approximately, that is, so they may afterwards be corrected and fixed by linear perspective. Next, the working-drawing plan may be put into perspective, assuming a station-point, a horizon, etc., in conformity with the above approximations, and placed, not as it is to stand in the finished drawing, but at any convenient distance below this position, or even above it. The reason for this is because, in a complicated subject, the mixture of superimposed lines near the level of the eye would give rise to confusion, and by lowering the plan the working-lines can be kept separated, and the intersections of the various measuring-lines with those of the plan will appear more sharply defined. As much or as little of the details may be filled in entirely freehand as the draftsman feels competent to execute correctly.

85. Illustrative Example.—The architect, as mentioned in the preceding paragraph, has recourse to the perspective plan a great deal, for, though architects have each their pet method of applying perspective principles, yet they are thrown in common upon their working drawings for data, and the use of the perspective plan, in some way or other, is probably quite universal. Let us consider this a moment a little in detail.

The architect has in front of him the working drawings of a building the final effect of which he desires to show to the public. He knows the point of view which is going to produce the best effect, but he cannot, without considerable wasted effort, attain this satisfactorily by assuming all of his perspective conditions. He makes then a sketch such as Fig. 51, which, perspectively, may be very incorrect, yet it shows him the general effect; this he proceeds to analyze and by means of it to locate the various perspective data; made in light pencilling, it can be erased as the more correct form is obtained.

Fig. 52 shows his final result. A portion of the ground-plan of the structure, with the superimposed roof-plan, is laid out at any convenient distance below the finished drawing,

FIG 51.



often upon a separate piece of paper which can later be removed; the further below the horizon it is placed, the more distinct and correct will likely be the various intersections. The figure does not show all of the data obtained, as the construction would be confusing; how carefully the features are worked out depends upon the degree of accuracy which is desired. After the ground-plan had been worked from, the second-floor plan could be superimposed, and if it were a very tall structure, this process could be carried through each story if there were changes sufficient to warrant it; the plans of the upper stories, moreover, might conveniently be put into perspective *above* the horizon. The upper part of Fig. 52 shows the finished product; the constructive work, which is very simple, and is mainly measurement by parallels in vertical planes, is not preserved. It can easily be seen that there is abundant opportunity in a drawing of this character for the artist to cut loose from the perspective construction and draw in various features by eye; indeed he *does* do this largely, the perspective only aiding in the proper placing of the salient features.

In drawing the perspective plan in this case just described, the picture-plane was assumed to go through the nearest edge of the house, so that the line *ab* is the scale of measures for horizontal distances. Whether the picture-plane be assumed as here or not is determined by convenience in construction. It might have been drawn, for example, to touch the corner of the porch instead; *a′b′* shows this position. If this had been done, then the walls of the house would have been continued forward to cut the picture-plane at *c* and *d*; hence all measurements of distances from the corner of the house along the plan would have had to have added to them the distances the corner was from the picture-plane measured along the line *ce* or *de* respectively. If there is an extension of but a slight feature beyond the front of the house, the position of the line *ab* is perhaps best as in this drawing; it is taken in this instance, however, to show that, whether here or elsewhere, the con-

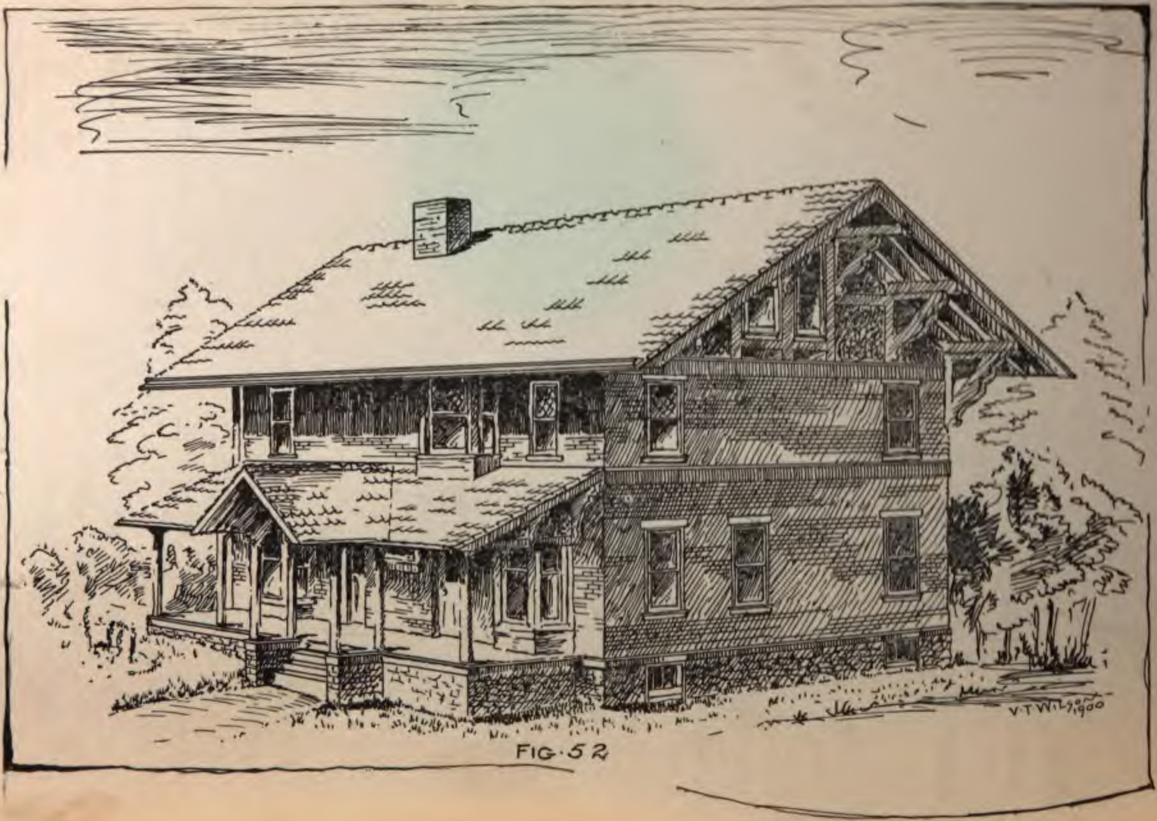
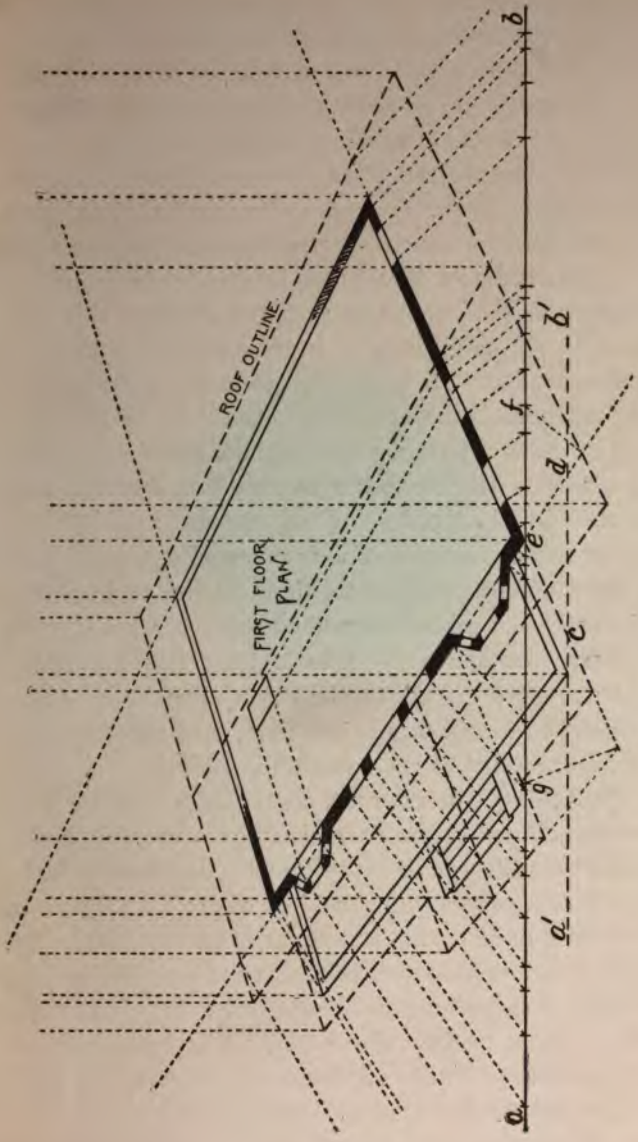


FIG. 52


ROOF OUTLINE
FIRST FLOOR PLAN
a
a′
g
c
e
d
f
b′
b

structive process is the same; the same measuring-point is used for the same parallel lines, the measurements being projected forward to cut the proper lines instead of drawn into the picture; *f* and *g* are preserved as showing such points for the construction of the main roof and the porch roof-line.

86. The Perspective Plan referred to the P.P. or to a Simple Figure.—Sometimes, in order to simplify the perspective construction as much as possible, the plan-view of the subject is referred to a line which is parallel to the P.P., and the distances various points of the plan are from this line, are measured on perpendiculars to it through the points; this, it can be seen, reduces all these distances to a geometrical basis of lines perpendicular and parallel to the P.P., and the measuring-lines of the first form the diagonals of squares. This reference to a line is similar to the method employed in the case of the circles in Fig. 29.

In case the drawing is of a number of separated subjects, it may be found convenient to circumscribe their plan-views collectively with a checkerwork of squares which, when the whole is put into perspective, have their edges parallel and perpendicular respectively to the P.P. The lines of the plan can be sketched in afterwards in reference to these squares and without exact regard to their vanishing-points.

87. Sketching Direct from Fact Data. — The practical problem sometimes arises to make, not a careful perspective drawing as that illustrated in section 85, but a sketch of more or less completeness of some subject where its form is known by a statement of fact; this statement of fact may be by written or verbal description of the subject or it may be in the well-known form of a working drawing, i.e., a drawing in projection where sizes can be represented by several views and in their proportional dimensional value. It has been thought best to give this subject separate consideration after the student has had an opportunity to benefit by all the perspective principles herein presented; Chapter XI will deal with it in detail.

88. Distortions.—It has been remarked before that, except in large extended subjects, distortion in a picture should not give very much trouble if the result aimed at is thoroughly understood. That it does occur when the observer is assumed too close to the picture-plane relative to the size of the object drawn was stated, and this is the chief element of difficulty. Of course in a subject of moderate size it may sometimes be necessary to introduce a perspective on the assumption of a far-distant observer, and at the same time develop details to an extent only possible to be seen if the subject is viewed from near by, but this is not really a matter of distortions, but one of treatment.

How far, it may be asked, can the observer be placed from the picture-plane to retain a natural effect? We frequently see it stated that naturalness in a picture is insured if all objects treated fall within a visual angle of 60°; this is, however, arbitrary and is only partially true; the full answer is that there is no limit. What settles the question in any case is whether the natural effect sought is really obtained. If, in making a drawing, a preliminary sketch is not used, but a station-point assumed arbitrarily, then the only way open is to see if the assumption results in what is wanted; if it does not, then the work has to be done over again with a new assumption and this process continued until the effect is right, and it must be so for an observer in any position relative to the picture. Drawing would be very much simplified indeed if it could be taken for granted that it will always be looked at from one point of view.

The effect, or natural appearance, is then the controlling condition in any case, not truthfulness to the principles of linear perspective. There are certain forms and effects upon which there is universal agreement, arising from custom perhaps more than anything else, and these are always expected in a drawing. The circle is one of the forms which undergoes modification, differing according to its associations in the picture. Then, again, every one expects to see groups of vertical lines drawn vertical and parallel as if the picture-plane were vertical, whereas if one were to stand near a tall building or a church-spire its lines would converge to a vanishing-point, and the observer would have to remove to too great a distance for clear vision before the lines ceased to do this quite perceptibly. Again, if a row of buildings whose front faces are in the same plane are viewed in a direction perpendicular to the row, it is required that they have their long horizontal edges parallel to the picture-plane with no vanishing-point, whereas if an observer were looking at these buildings themselves, it would be impossible to see all without turning the head, and as soon as that was done convergence of the horizontal lines would be the apparent result. These and other effects are demanded by custom.

The last-mentioned case illustrates quite a common experience in large subjects. If a camera were used to make the picture, it would be shifted and a succession of views taken which would be mounted side by side. This gives a very good effect, and in general distortions, so called, arising from it are not offensive to the eye.* This is in reality making a picture on a succession of flat planes, forming a prism, with the imaginary station-point on the axis. This method may be sometimes used in a drawing, but perhaps the more common plan is to draw the row of buildings, previously mentioned, in parallel perspective and introduce minor and detached details as if they were drawn when viewed separately, the station-point assumed shifted horizontally for the purpose opposite each detail in turn.

In the small drawings, made in a classroom, of groups of geometrical solids, it may truthfully be said that the process is mainly drawing on the inner wall of a cylinder, the drawing-paper being this wall unrolled; the distance of the observer from the picture-plane and the small area of the wall included in the picture render the changes from the flat picture-plane method unnoticeable. If, in holding up a pencil or thread to get relative directions of edges, the student comes across irreconcilable facts, they may often be ascribed to this peculiarity of his drawing.

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* It may be pertinently suggested that the distortions would be less criticised in a photograph, which most people believe is faultless, than they would be were they in a drawing.

The constantly shifting actual observer requires the draftsman to use license in the application of perspective rules, and he is conceded this license by custom without comment. It is only when a hypercritical observer comes along to find faults with his work on the basis of scientific perspective that any trouble arises. Perspective is the artist's help, not his hindrance, and should always be used in that way, and to enable him to get an effect which seems correct no matter from what point of view it may be observed.

CHAPTER VIII.

CHANGES IN FORM DUE TO CHANGES OF POSITION AND THE ANALYSIS OF THE PERSPECTIVE OF CERTAIN GEOMETRICAL SOLIDS.

89. General Statement.—The subject of the relative foreshortening of lines and surfaces, when they have been required in problems, has been purposely deferred until a fuller comprehension of perspective principles has been had. It is now proposed to take the matter up in connection with a very instructive, and sometimes important, phase of perspective, namely, the changes in appearance which forms undergo when their position is changed, and this investigation will be begun by the study of simple groups of lines moving in different directions relatively to the picture-plane.

90. A Moving Line Generates a Plane.—If a line is conceived to move in space, it generates a surface of which it is the successive elements; if parallel to itself, then perspectively the vanishing-point of the moving line remains fixed.

The surface may be flat or plane if the direction of motion is constant, or it may be curved if the direction of motion is changing. If the rate of motion is different for different parts of the line, as in the directed plane described in section 71, then the vanishing-point of the moving line is a constantly changing one; in fact the vanishing-point has as its locus, in successive positions, a line on the picture-plane which is the vanishing-trace of the plane generated by the moving line. The vanishing-trace will have a direction on the picture-plane which is determined by the direction of motion of the line; for example, if the line described in section 71 were indefinite in length, swinging in a plane about a point, it will have for its vanishing-trace that line on the picture-plane which is found by passing a plane through the station-point parallel to the plane generated by the line. Or again, if the line move in space so that it generates a circular cone, then the locus of the vanishing-points of the line will be found in the line of intersection of a similar cone with the picture-plane, whose apex is at the station-point. The vanishing-trace may, therefore, be any one of the conic sections, depending upon the relation of the axis of the cone generated to the picture-plane.

91. Rotation of a Horizontal Square Plane about a Vertical Axis.—To make a practical application of this study of the motion of a line, consider the motion of a group of lines forming a square plane, first horizontally as shown in Fig. 53, and see what perspective changes can be noticed. Suppose *aa'*, the axis of the square, is also the axis of rotation, and its direction that shown by the arrow. The vanishing-points of *AB* and *CD* are infinitely distant to the right and left. As the square is moved a vanishing-point appears on the horizon, at the right, a finite but long distance off. At the same time the other vanishing-points move towards the left. If one quarter of a full revolution is considered, then, as the vanishing-point of *AB* and *CD* moves from infinity to the position *V.P.*[1], the latter has only moved the distance from its former position to *C.V.*; likewise the vanishing-point which was at *C.V.* moves to *V.P.*[2], and the latter to infinity to the left. The edge *AB* decreases in length through all values to that of *BC*, and similar changes are seen to take place with the other edges. The shortest length which an edge of the square can appear is that of *BC* in Fig. 53.

If only one-eighth of a complete revolution is considered, then the position of the square is that shown in Fig. 54; one diagonal is parallel to the picture-plane, and the other perpendicular to it. The general area of the square in any of its positions does not change appreciably, and the length *AC* of

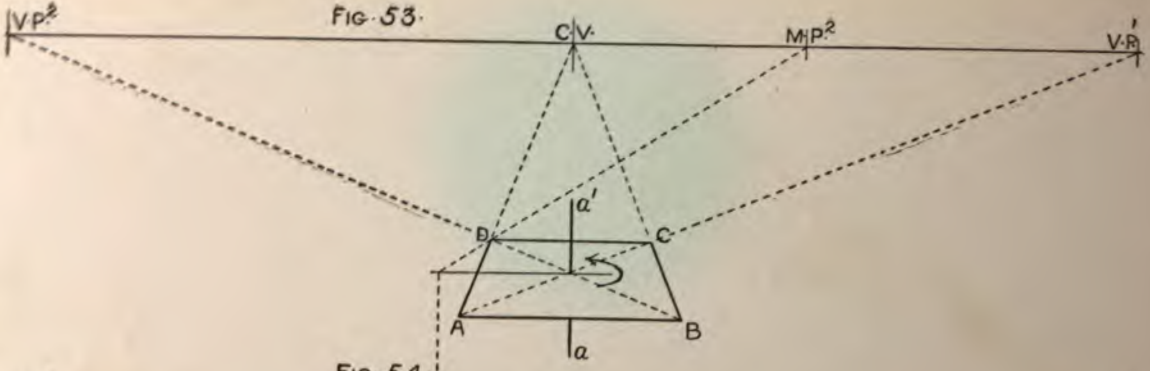


FIG. 53.

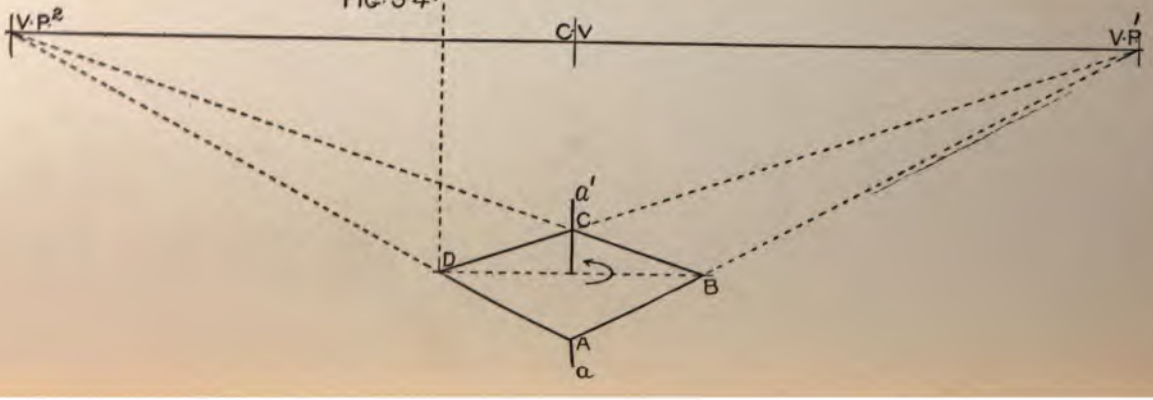


FIG. 54.

Fig. 54 is slightly, though not much, greater than the perpendicular distance between AB and DC of Fig. 53; its difference depends upon the level of the eye with respect to the plane of the square, being greater as the eye is raised.

Before leaving this example it will be interesting to inquire what the length AB of Fig. 54 should be relative to its true length; it is not possible to decide with definiteness without perspective construction, but some hint of how to estimate it may be obtained, and the principle involved may be applied to any line no matter what its position relative to the picture-plane. If the square be conceived to move uniformly to the position of Fig. 54, then the edge AB will undergo foreshortening, at first very slowly but more and more rapidly, so that AB of Fig. 54 is not midway in size between AB and BC of Fig. 53, but rather greater. The maximum limiting value, which, however, it could not reach, is the side of a 45° triangle of which the true length AB of Fig. 53 is a hypothenuse. In like manner the foreshortened value of any line can be estimated. For if the line can be conceived as parallel to the picture-plane, then rotated into its position in a plane perpendicular to the picture-plane, a suggestion may be obtained of its probable length in the position desired. See also section 76.

92. **Rotation of a Square on an Axis Perpendicular to the Picture-plane.**—Next consider the same square as rotated about a line which is perpendicular to the picture-plane, and for convenience take a diameter bb' as shown in Fig. 55. Several positions of the square are drawn, and give some idea of the relative areas of the plane, which change from the value $ABCD$ to zero during one-quarter of a revolution, together with other modifications. Here the edges BC and AD move parallel to themselves and consequently $C.V.$ remains their vanishing-point; likewise AB and DC continue in all positions to be parallel to each other and to the picture-plane. The fact that is chiefly interesting in this rotation is that the diagonals move about the axis, maintaining a constant angle with the

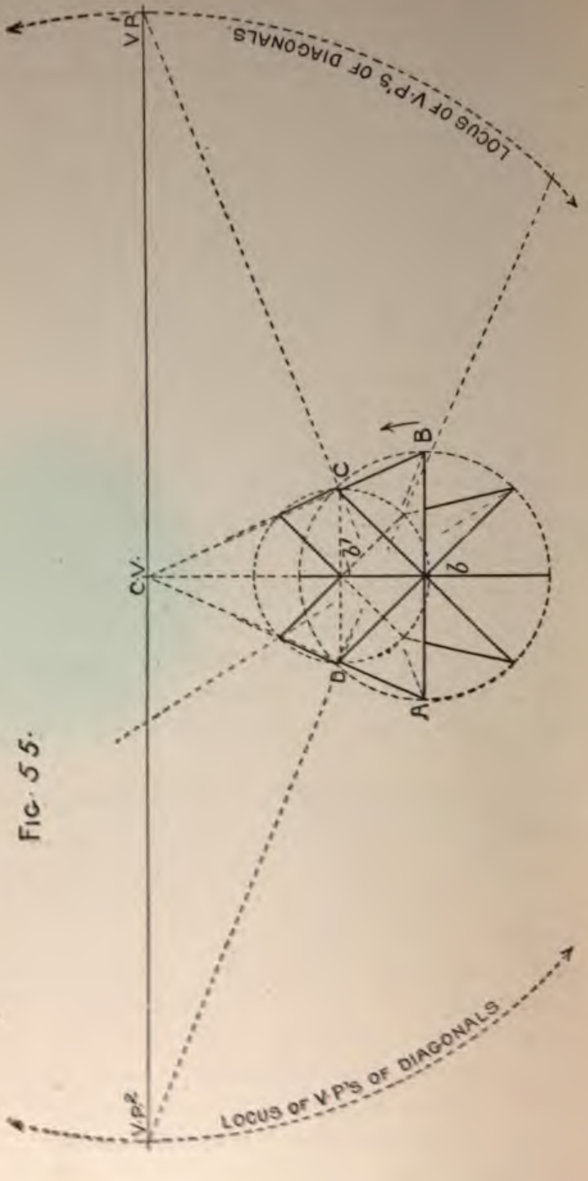

V.P.¹
LOCUS OF V.P.'S OF DIAGONALS
C.V.
C
B
D
A
b
b¹
V.P.²
LOCUS OF V.P.'S OF DIAGONALS

FIG. 55.

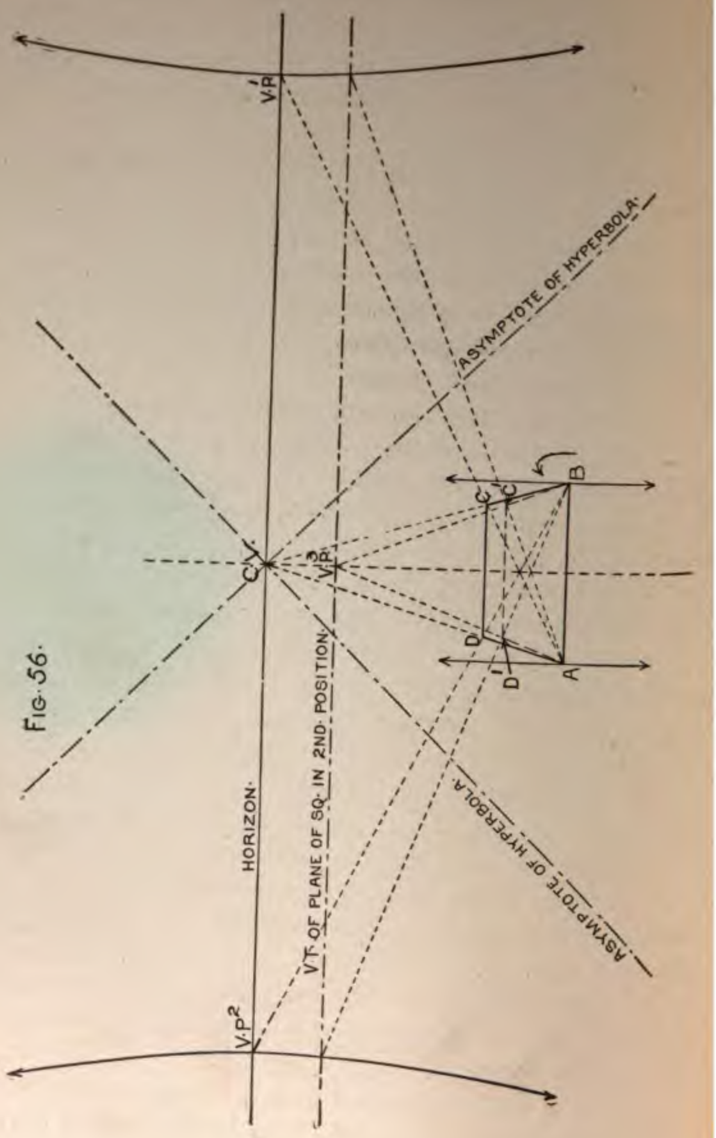
picture-plane, generating theoretically a conical surface, and of course a similar cone and similarly placed, having its apex at the station-point, will intersect the picture-plane in a circle, with *C. V.* as its center, which will be the locus of the vanishing-points of these diagonals.

If the square were rotated from the position in Fig. 54 about the diagonal which is perpendicular to the picture-plane, then a circle corresponding to this one becomes the locus of the vanishing-points of the receding edges of the square.

93. Rotation of a Square on a Horizontal Axis which is Parallel to the Picture-plane.—Perhaps the most instructive facts concerning the changes of form of the square can be obtained by raising or lowering the square in a motion which is parallel to itself, or by rotating it about an axis which is parallel to the picture-plane. As the latter embodies the principal changes to be noticed in the former, it will only be necessary to discuss this one fully. For convenience, assume the axis of rotation to be the side *AB* which is illustrated in Fig. 56, and the direction that shown by the arrow.

The initial or first position is that shown at *ABCD.* If moved about *AB* downward, the edges *BC* and *AD* describe vertical planes, hence their vanishing-point moves along a vertical line through *C. V.*; *V.P.*^{3} is its position after the square has moved through a certain angle; when the square becomes parallel to the P.P. this point will be at infinity below. The plane undergoes change of foreshortening as the angle with the P.P. changes. The vanishing-trace of the plane of the square will in all positions be a horizontal line; hence a square plane sketched as in the position *ABC'D'* without other accessories might or might not stand for a horizontal plane. The presence of a little shadow or association with other forms is required to show its true position.

To those who are familiar with the geometry of conic sections it is instructive to observe that the diagonals of the square in the rotation describe conical surfaces with apexes at *A* and *B* and axes parallel to the P.P. Therefore, in this case, as in



FIG·56·

HORIZON·
V·P¹
V·P²
V·P³
C·V·
V·T· OF PLANE OF SQ· IN 2ND· POSITION·
ASYMPTOTE OF HYPERBOLA·
ASYMPTOTE OF HYPERBOLA·
A
B
C
C¹
D
D¹

preceding example, if a cone be conceived to have its nents in the direction, in their successive positions, of these onals and its apex at the $S.P.$, then it will be a double cone intersect the picture-plane in curves through $V.P.^1$ and $.^2$, which are the curves of an equilateral hyperbola of two bes, having as asymptotes lines in the direction of the onals of the square when the latter is parallel to the P.P., h last is seen to agree with the conditions of the rotation.

f the square plane be conceived to be moved parallel to , the vanishing-points remain unchanged, and the only ences are that the apparent area changes and also the rate nvergence of the receding edges.

4. Changes in Form in a Solid in Different Positions.— above illustrations serve to indicate how position affects form of a simple geometrical figure, and they show how stigation may be profitably carried on in the case of any ds as well as plane figures. Many common forms can be ced to a combination of those geometrical solids with which draftsmen are familiar, and in connection with this con-ration of change of form it is proposed to make a brief mary of these and discuss their possible perspective forms.

With this knowledge ready at hand and an application of directions given in section 86, concerning the analysis of a ect by division into simple circumscribing figures, the tsman will be materially assisted in the interpreting of forms n they are not present to draw from.

95. Perspective Possibilities of the Solids: the Cube.— cube is the simplest of the geometrical solids. Its drawing n the station-point is in the plane of the top and when it is ding on a horizontal plane in several different positions is wn in Fig. 57.

The position *a* is when the line of direction from the ion-point to the center of vision is perpendicular to a verti-face. If the cube is very large or the station-point very se, in model-drawing it might be possible, by using the test ntioned for getting directions, to note convergence in the horizontal edges if the eye were opposite any point but the center of the face, but they would not be drawn to show this any more than convergence in the vertical edges would be shown. In *b* the cube is turned horizontally through a very slight angle; the vanishing-point of the edges on the least visible face is very near, hence the lower edge inclines at a very large apparent angle to the P.P. In this and the succeeding cases the C.V. is supposed to be somewhere within the limits bounded by the vertical edges of the cube.

FIG 57

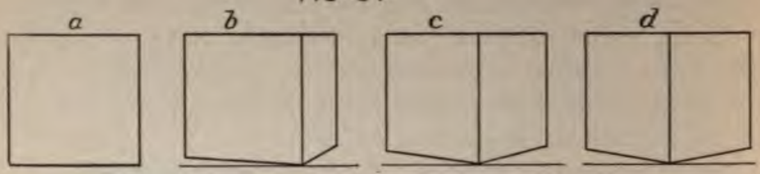
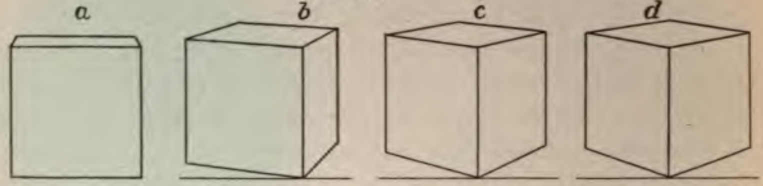


FIG 58



In *c* it has moved still further around; inclination on the left-hand edge is greater than before and on the right-hand one less, until if moved to position *d* both are equally inclined, and both vertical faces are equal in area. This is the position of maximum apparent width of the cube, and *a* that of the least. These changes can best be understood by reference to the figures of Fig. 58, where the eye is supposed to be above the top face of the cube. The various inclinations of the edges relative to each other are seen more clearly; the diagonals of the top face help to show this. In *a* the diagonals are equally inclined, while in *d* they are perpendicular to each other. The effect in the different positions, when the eye is in the plane or below the bottom face of the cube, can be seen by turning Figs. 57 and 58 respectively upside down. A position of the cube which is often met with is shown in Fig. 59. The name 'symmetrical position' may be given to it, as it is of use and valuable to remember. It gives the maximum total area of the solid possible in a drawing. The line from the nearest corner to the farthest coincides with the line of sight, and one diagonal plane of the cube *ab* is viewed tangentially, hence contains the vanishing-point of three of the receding edges. The vanishing-points of the other receding edges are equal distances either side of it, in some line or trace perpendicular to it. The faces 2 and 3 are equally inclined to the P.P., and the inclination to the P.P. also of the plane of the face 1 is slightly greater than that of the edge *ca* and those parallel to it. If the line of intersection with the P.P. of the face *l*, in any position of the cube, were drawn, it would be a line parallel to the diagonal *lf*. The drawings show how varied may be the positions of this view with respect to the P.P.

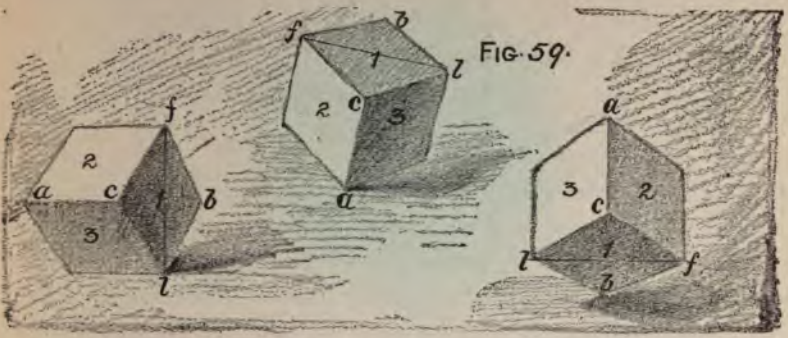
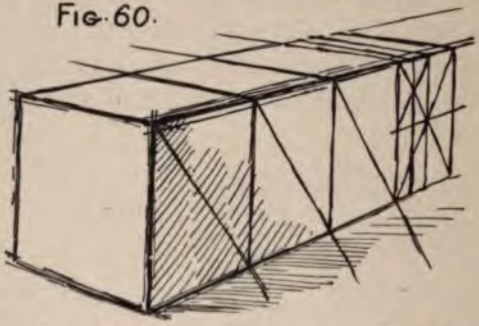


FIG. 59.

To give the proper value, in first laying out the cube in this last figure, where the faces are all inclined to the P.P., it might be of assistance to circumscribe the face 1, for example, with a square, the middle of whose sides touched the diagonals of this face. This brings the square into a position in which there is only one direction of inclination to observe, and it may be more easily drawn.

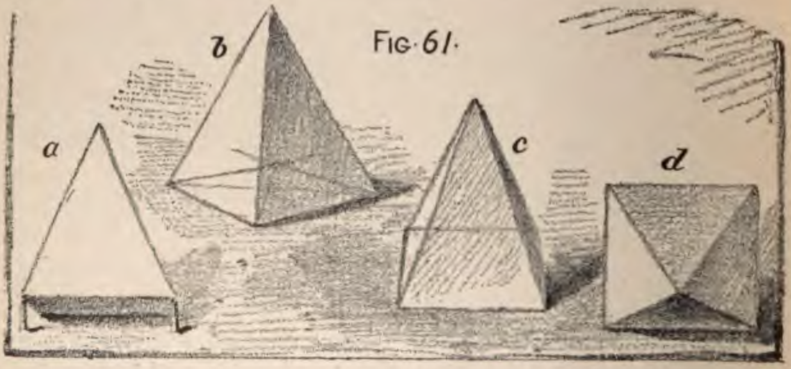
Many forms met with may be analyzed by using the cube as a unit; the square prism, for example, is one of them. It need hardly be discussed by itself; it can be built up on a basis of cubes side by side, even if its longer faces are not divisible into an even number of squares. By using diagonals and continuous subdivisions (see section 50) any fraction of an addition to an even number can be easily drawn. Fig. 60 shows such a solid with the constructive sketch-lines preserved.

FIG. 60.



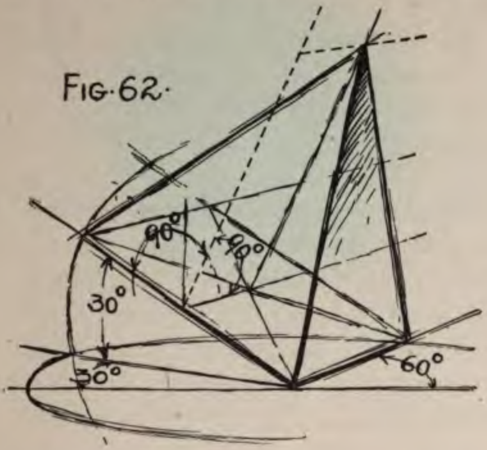
96. The Square Pyramid.—The square pyramid consists of a square base whose perspective properties have already been discussed, and four triangular faces meeting at the apex. It may occupy positions in which one, two, three, or four triangular faces are visible. Fig. 61 shows these positions. The length and direction of the axis for any given pyramid is the only feature needing special comment. As this is perpendicular

FIG. 61.



to the base, at its center, its apparent or foreshortened length will vary inversely as that of the base; after this is drawn, lines connecting the apex with the corners of the base will complete the pyramid and by their position determine how many faces are visible. The axis may be located conveniently by estimating the angle of 90° with the base in any chosen position; a plane can be sketched passing through and containing the axis and a diameter of the square base which may be made to represent the directed plane of rotation of a definite line, or the plane may be constructed perpendicular to the base and through an edge, and the axis directed and cut off by means of parallels;

FIG. 62.



this construction is shown in Fig. 62. In the absence of any knowledge of the relative foreshortening of the axis, and any edge of the square (it should be compared, for greater convenience, with one least foreshortened), it may be arrived at by conceiving the axis and the edge so placed as to form the edges of a rectangular plane and this plane determined in its proportions by analysis into squares and fractions of squares, as the face of the prism is in Fig. 60. If it were required to draw the pyramid by the relation of a triangular face with the base, a diagram could be drawn and the angle of one to the other compared with the well-known angles 30°, 45°, and 60°.

An example of this would be when it was desired to sketch a pyramid having certain proportions, and one triangular face horizontal.

97. The Hexagonal Plane and Figures containing it.—The hexagonal plane should first be analyzed as to certain of its geometrical properties, which are: (1) It is divisible into six equilateral triangles; (2) The diameter as compared with a side or a diagonal is incommensurable; (3) Its diagonal is twice the length of a side; (4) The opposite sides are parallel respectively two and two; (5) Any diagonal* is parallel to and equidistant from the opposite sides; (6) The lines from the extremities and middle of opposite sides divide the diagonal they cross into four equal parts; (7) It can, like other regular polygons, be circumscribed by a circle, the diameter of which is equal to the diagonal of the hexagon, or it may be circumscribed about a circle, in which case the diameters of each are equal.

These properties are preserved perspectively, the various features undergoing foreshortening, of course, when inclined to the picture-plane. In the case of the lines from the extremities and middle of opposite sides dividing the diagonal they cross into four equal spaces, it may be said that, when the hexagon forms but a small feature in a drawing, as, for example, when it goes to form a nut on a tool or piece of machinery, the foreshortening of the divisions is a negligible quantity, for the differences would be too slight to record.

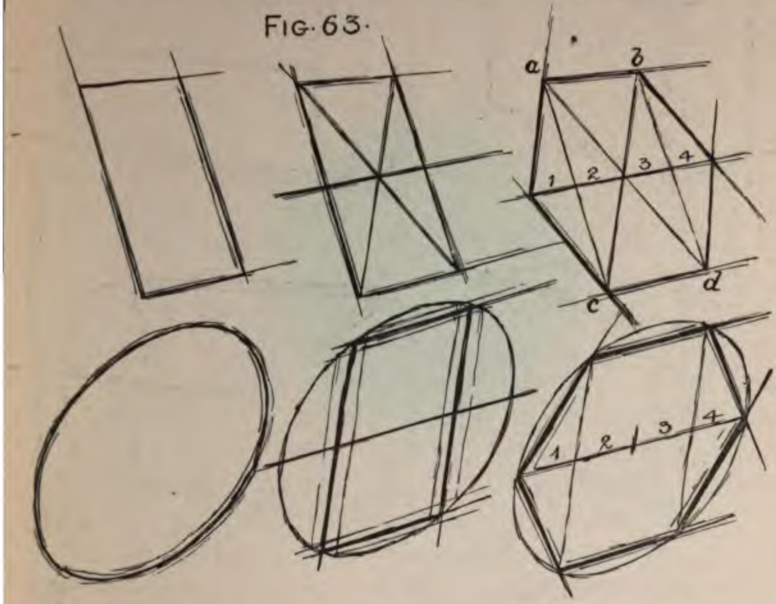
The hexagon may be sketched by the aid of the above properties in two general ways, as shown in Fig. 63: first, by using the rectangle *abcd* and the four equal divisions 1, 2, 3, 4, etc.; and second, by drawing the circumscribing circle, then inscribing the hexagon in it. This last is exceedingly useful, although it involves considerable adjusting by successive approximations to get the proportions correctly. As in the case of a square circumscribing a circle, the hexagon can be situated variously within a given circle. Any side, diagonal, or diameter can be used from which to start the sketch, and any direction, moreover, can be given to this first line.

---

* 'Any diagonal' is a term here used to mean the longest diagonal that can be drawn.

The Hexagonal Prism is one of the forms containing the hexagon which illustrates how convenient is this last method. The prism in the form of a nut or bolt-head, as part of a drawing of a piece of machinery, for example, may be first circumscribed by a cylinder of such proportions that the prism, of a certain size, can be inscribed within it. The two methods of building up the prism are similar to those in the case of the plane, however, and are shown in Fig. 64.

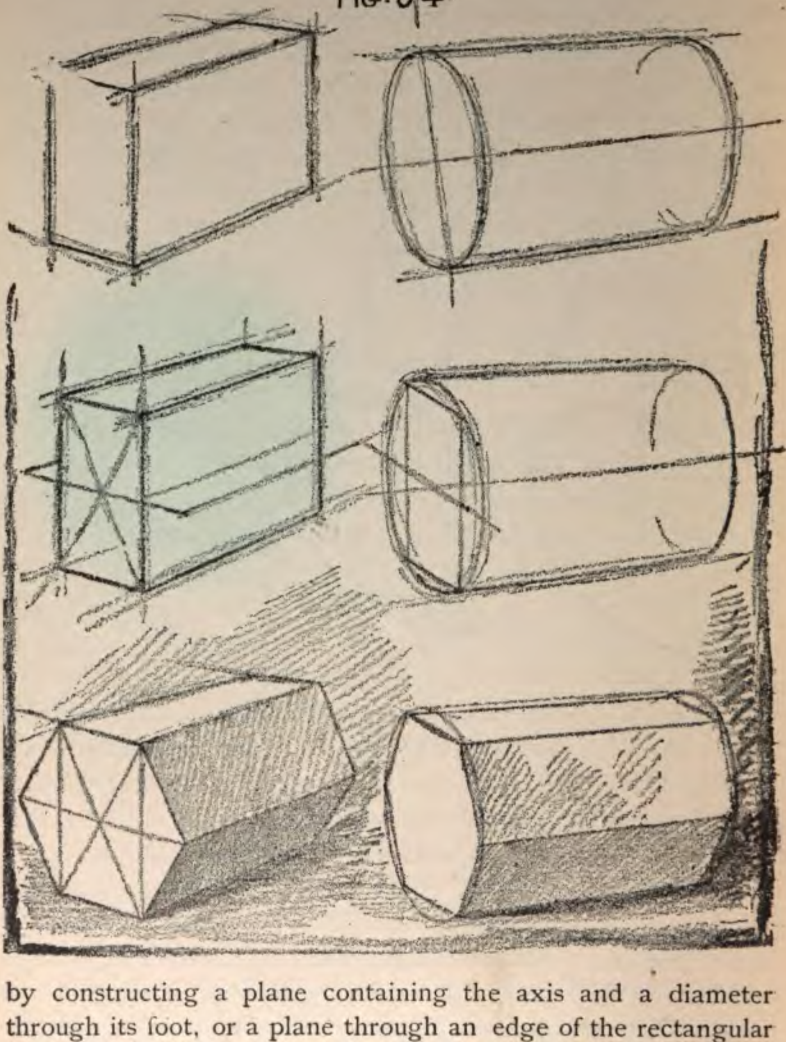
FIG. 63.



When the prism is so situated that three rectangular faces are visible, two may be of equal area, in which case they are the outer faces; the middle one will be a rectangle without foreshortening. The line of sight will be perpendicular to this face.

The Hexagonal Pyramid may be built up in a manner similar to the square pyramid. In fact all the pyramidal forms can be treated alike. This also can sometimes be conveniently inscribed in a circular cone whose axis, of course, will be superimposed upon the minor axis of the ellipse of the base. The direction of the axis of the pyramid, when the circumscribing cone is not used, may be obtained, as in the square pyramid, by constructing a plane containing the axis and a diameter through its foot, or a plane through an edge of the rectangular figure, as *ac*, Fig. 63. The length of axis relative to the size of the base and the amount which is seen of the base govern

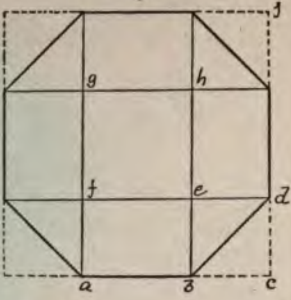
FIG. 64.



the number of faces which can be visible. In an ordinary pyramidal form which is a small feature of a drawing, it may be observed that, if viewed from such a point that a line of direction from the station-point through the apex falls within the plane of the base, the latter does not undergo foreshortening to a degree to warrant any effort to represent it; the differences are practically negligible.

98. The Octagonal Plane and Figures containing it.—The octagonal plane possesses a few geometrical properties which perhaps can be pointed out best by reference to Fig. 65. (1) The intersecting diameters form a square, as *efgh*. (2) The octagon can be circumscribed by a square and also a circle. (3) A side, as *cj*, of the circumscribing square is divided into three parts, the extremes bearing that relation, as to size, to the middle one that the side of a square does to its diagonal. (4) The exterior angles are 45°.

Fig 65



The fact that it can be circumscribed by a square suggests a way to treat it, and the perspective properties may be summarized as those that apply equally to the square prism.

The Octagonal Pyramid can likewise be built up on the basis of a square pyramid and then the necessary modifications introduced to turn it into the octagonal. In inscribing it in the square pyramid there is this advantage over the hexagon which is inscribed in a cone: little or no allowance has to be made for difference in size between the two, as has to be done in the latter case.

99. The Cylinder and Cone.—The cylinder has already, in part, been discussed in connection with the circle in Chapter V; only one or two points remain to be touched upon. If the line of sight is perpendicular to either base, the opposite one is hidden by the wall of the cylinder, approximately one-half of its perimeter being visible. If no portion of the wall of the cylinder appears, or the bases are parallel to the P.P., the visible base should be drawn a true circle. It is convenient in some cases to locate a cylinder by first circumscribing it with a square prism. Such a drawing of a cylinder is shown in Fig. 66*a*; it is lying on an element on a horizontal plane, its axis at any convenient angle to the picture-plane. It is here given in order to point out the additional facts that the dotted line through a shows the element touching the horizontal plane, and that bb' is vertically over it and represents the highest element of the cylinder. cc' is one element cut out by a horizontal plane passed through the axis. Fig. 66*b* shows what these highest, lowest, and middle elements become when the plane of a base is placed parallel to the picture-plane.

FIG. 66*a*

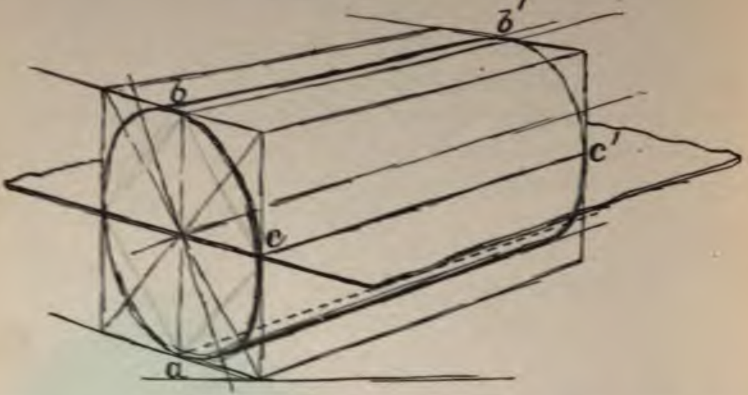
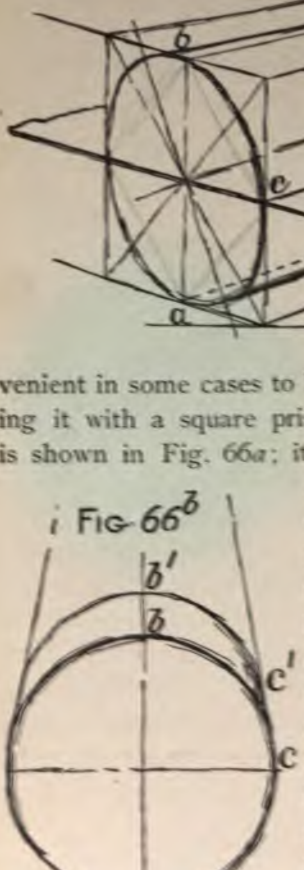


FIG. 66*b*



The Circular Cone deserves some attention from the fact that it is quite a common form. The minor axis of the ellipse representing the base is always drawn as a continuation of the axis of the cone, and, as mentioned in connection with the square pyramid, the foreshortening of the axis is inversely as that of the base; and it is also to be noted that when the line of sight lies within the plane of the base, the contour elements of the cone will have their greatest apparent length, and as the cone is moved from this position or the axis is turned at increasing angles to the picture-plane, the visible contour elements change, becoming shorter lines, from the apex, *always tangent to the perimeter of the base*, however, and at equal distances from the extremities of its major axis. Several of these positions are shown in Fig. 67. If the line of sight lies within the plane of the base, the cone becomes an isosceles triangle; if the apex falls within the perimeter of the base, the latter is always drawn a true circle, no contour elements are visible, and the appearance of a conical surface, if the apex be turned

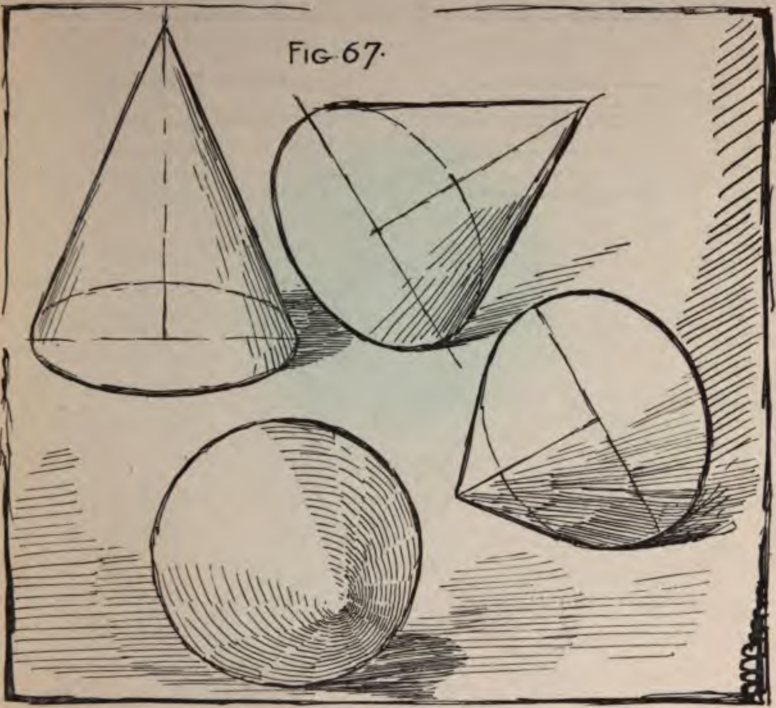
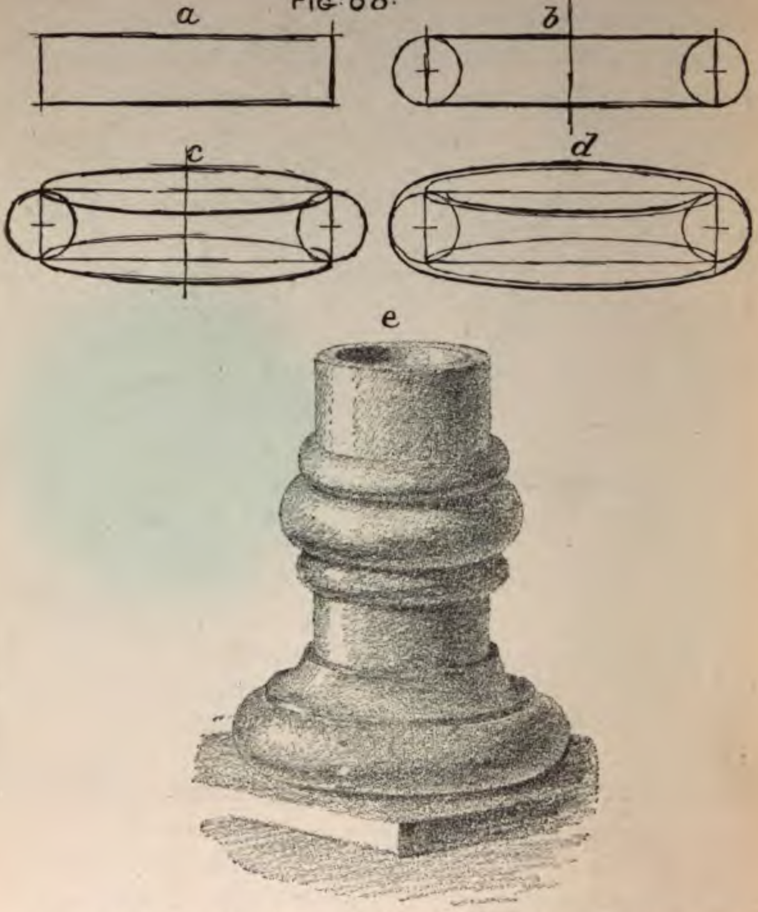


FIG. 67.

toward the observer, can only be obtained effectively by the use of shadow. In any position, the angle which the axis makes with the base can be constructed as described in connection with the square and hexagonal pyramids.

Fig. 68.



100. A Ring Circular in Cross-section.—This is a form which may not be recognized at first as a type of forms commonly met with. But it is, however, a convenient basis for the consideration of certain forms known in architecture as the torus, and also what is known as a half-round, or bead in moulding. The ring is shown at *d*, Fig. 68. It is not a form bounded by an elliptical curve, as might at first be supposed; a consideration of *a*, *b*, *c*, and *d* will show how it may be built up constructively upon elliptical and circular forms, and that its true curve approaches to these, but varies somewhat according to the point of view taken. The common form which this curve takes is that shown at *e*. From one position, of course, the ring will appear as two concentric circles; from a position at right angles to that it will appear two parallel lines limited at their ends by circular arcs.

Notice that at *d* the contour curve is tangent to the base circles at the right and left; this is true in all positions, and that it lies near but is not tangent to the base ellipses at the highest and lowest points of the figure; this is also true in all positions except the latter one of those mentioned in the preceding paragraph. The degree of proximity depends upon the angle at which the plane of the top and bottom is viewed. It recedes from the base ellipses as this angle increases.

101. **Summing up.**—A number of the type forms have thus been analyzed, their geometrical and perspective features discussed, and it has been also indicated how, in the absence of models, these forms of any desired proportions may be sketched by a constructive process, building up part by part according to the perspective properties which are most easily handled, and one form referred to another if the latter can be more readily drawn than the former. When the student comes to combine these forms into an intricate subject, then the value of these constructive processes can be better appreciated, and such a combination of them is treated of in Chapter XI. Before attempting this, however, it is well to attain some facility in sketching the simpler forms and estimating fore-shortening, and the value of angles and the best analysis to use for this purpose.

102. **Problems.**—Following are a number of subjects for practice, described as to their forms; perspective sketches are to be made of them under the conditions named. The dimensions are given in units so that the sizes of the drawings can be

varied to suit, and unless otherwise specified it is to be understood that in each case the entire subject is to be below the level of the eye, but how much is left optional.

Problem 1. Sketch a cube of unit edge standing on the horizontal plane, its vertical faces at angles of 20° and 70° respectively to the P.P.; from the middle of each face extend cylinders one-half unit in diameter, and length one unit; the axis in each case is to be perpendicular to the face of the cube.

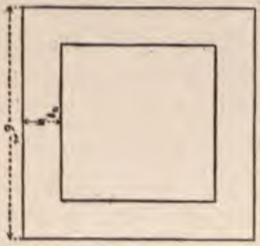
Problem 2. Sketch a cube of unit edge standing on the horizontal plane, with vertical faces at 30° and 60° to the P.P. From the middle of each face extends a circular cone, axes perpendicular to the faces, bases of cones three-quarters of a unit in diameter, length of axes the same.

Problem 3. Sketch a rectangular prism, one unit square by four units long; stand on an edge so that the square face adjacent is at an angle of 45° to the horizontal plane, and two rectangular faces at an angle of 45° to the P.P.

Problem 4. Sketch a square pyramid, two units on edge of base, and three units altitude; stand so that the plane of the base is at an angle of 30° to the horizontal plane; other conditions optional. Support it conveniently in its inclined position by a vertical rod resting with one end upon the horizontal plane upon which the pyramid is standing.

Problem 5. Sketch a square pyramid, proportions optional; place it on one of its triangular faces on the horizontal plane, the altitude line of this face at an angle of 45° to the P.P.

Fig 69



Problem 6. Sketch a square frame such as is shown in Fig. 69 and with the proportions as marked, standing on a long edge which is at 30° to the P.P., and the plane of the adjacent square face at 45° to the horizontal plane.

Problem 7. Sketch a series of five equal hexagonal planes so situated that they have a common edge in the horizontal plane, which edge is at an angle of 45° to the P.P.; two of the planes are horizontal, one vertical, while the remaining two stand at angles of 60° and 40° to the horizontal plane, either side of the vertical.

Problem 8. Sketch a hexagonal prism, diagonal of base two units, length of axis four and one-quarter units, standing on the horizontal plane on a rectangular face, the edges of the latter equally inclined to the P.P.

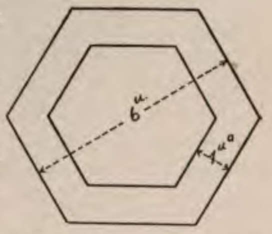
Problem 9. Sketch a hexagonal prism, diagonal of base two units long, length of axis three units, standing on an edge (of its base) which is at 20° to the P.P. and its rectangular face making with the horizontal plane an angle of 30° extending toward the P.P.

Problem 10. Sketch a hexagonal pyramid, two units on edge of base and five units altitude, standing on an edge of the base, which is at 30° to the P.P. The axis of the pyramid makes an angle with the horizontal plane of 20° and the apex is supported by a vertical equilateral triangular card.

Problem 11. Sketch a hexagonal frame such as is shown in Fig. 70, with proportions as marked, lying on a hexagonal face below the eye, edges of the face at any chosen angle to the P.P.

Problem 12. Sketch a hexagonal frame such as is shown in Fig. 70, with proportions as marked, standing on a rectangular face, with the edges of that face equally inclined to the P.P.

Fig. 70

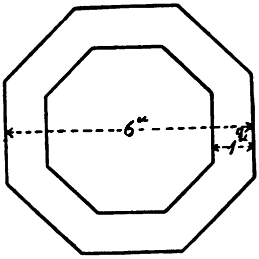


Problem 13. Sketch an equilateral triangular prism, one unit on triangular face and five units long; rest it on a short edge which is at 60° to the P.P., and the rectangular face adjacent to it at 30° to the horizontal plane.

Problem 14. Sketch an octagonal plane standing vertically on an edge, which latter makes an angle of 30° with the P.P.

Problem 15. Sketch an octagonal frame such as is shown in Fig. 71, with the proportions as marked; stand it on one of its longer edges, the adjacent octagonal face at 30° to the horizontal plane. Support it by a vertical rod touching the middle of the inside edge of the frame.

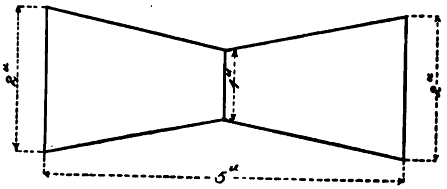
Fig. 71



Problem 16. Sketch a circular plate of five units diameter and one unit thick touching the horizontal plane at one point with the plane of the adjacent circular face at about 20° to it; support it by a vertical rod.* In the middle of the top surface of the plane place a hexagonal bolt-head, three units in diameter and one and one-quarter units high. Use an inscribed or circumscribed cylinder as a basis for it.

Problem 17. Sketch a cone, of two units diameter of base and five units altitude, standing on its base; about it place four other cones of same proportions lying on their elements, their apexes touching the base of the first cone, the elements of contact with the horizontal plane of two of the cones to be at zero degrees with the P.P.† on either side of the first cone, while those of the remaining two make angles of 60° and 45° respectively with the P.P. similarly on either side.

Fig. 72



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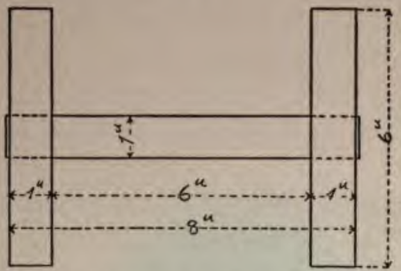
* Note that to have equilibrium the rod must support at a particular point.

† These two may be regarded as in parallel perspective with the planes of the bases perpendicular to the P.P. either side of the C.V. and drawing rectified as directed in section 66.

It would be well in this drawing to show a slight difference between the location of the centers of the larger and smaller ellipses.

Problem 18. Sketch a double circular cone such as is shown diagrammatically in Fig. 72, situated in several different positions with its axis horizontal, and at 30°, 90°, 45°, with the P.P. respectively.

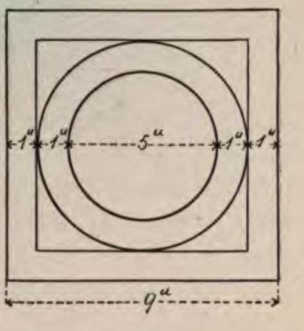
Fig 73



Problem 19. Sketch a spool such as is shown at Fig. 73, consisting of cylindrical disks and shank, and with proportions as marked; stand it so that its shank is horizontal and at an angle of 30° to the P.P.

Problem 20. Sketch an object such as is shown in Fig. 74, with proportions as marked, standing on a corner, the square face at an angle of 45° to the horizontal plane and its edges equally inclined to it. Other conditions optional.

Fig. 74.



CHAPTER IX.

PROBLEMS IN TANGENT AND INTERSECTING PLANES.

103. A General Statement.—It is now proposed to discuss the subject of plane surfaces more analytically and comprehensively than was done in Chapter II, to ascertain how to show their perspectives under any given conditions, for, while a careful analysis may not have been necessary for the purposes of outline sketching, it is of importance when studying the subject of cast shadows, as may readily be seen by considering a simple case. A line of cast shadow may be defined as the line of intersection of a plane passed through the line casting the shadow, and the surface receiving it; it is at once important to be able to pass a plane of any required direction through the given line, and find its intersection with any other desired surface, or, if the direction of the light is given, to know just what direction the plane of the shadow would take.

104. The Various Traces of a Plane.—In what has gone before but slight emphasis was laid upon any other trace of a plane than its vanishing-trace, while, in order to be able to locate any oblique plane, it is necessary to consider certain other traces to be discussed shortly. Unless a plane is vertical or horizontal, it must always be referred to two other planes or lines already known, before its exact position can be given. The same may be said of a line. In descriptive geometry, use is made for this purpose of a vertical and a horizontal reference plane, together called the coordinate planes of projection; in perspective two similar planes are used, namely, the picture-plane and some horizontal plane.

'*The Vanishing-trace of a Plane*' has already been defined as the picture of its infinitely distant limiting edge, and is found theoretically by passing a plane through the S.P. parallel to the given plane and finding the line of intersection of this plane with the P.P. This, it must be understood, is an auxiliary plane conceived to be drawn in order to ascertain the vanishing-trace. All other planes actually parallel to this one will have their vanishing edges identical with its trace, and, of course, all lines that might be drawn in them will have their vanishing-points in it also.

'*The Initial Trace of a Plane*' has also been defined as the line of intersection of a plane with the P.P. or with a plane parallel to the P.P. The initial and vanishing-traces of the auxiliary plane described in the preceding paragraph are identical. Planes parallel to it, while having their vanishing-traces identical with it, will have their initial traces in lines which must, by construction, be parallel to the vanishing-trace and at distances from it determined by the distances the planes are from the auxiliary plane. As has before been observed, the greater the distances from this auxiliary plane, the more of the plane surfaces would be visible to the observer. It also deserves to be recalled to mind in this connection, that these traces do not indicate by their direction the angles which the planes are with respect to both the horizontal plane and the P.P., unless they are perpendicular to the latter, in which case the vanishing-traces go through the C.V., and define by their angles with the horizon the exact positions of the planes.

'*The Horizontal Trace of a Plane*' is the line of intersection of a plane with any horizontal plane. Hence, if from any point in the initial trace of a plane, a line is drawn to the point in which its vanishing-trace intersects the horizon, then this line will be the picture of a horizontal trace; i.e., the line of intersection of some one horizontal plane with the given plane. It has often been customary in perspective treatises to speak of *the* horizontal plane, meaning thereby the surface of the floor or ground upon which the observer is standing. This is a restricted meaning; the horizontal trace, above referred to, can apply to any horizontal plane that it is desired to construct, whether above or below the observer's eye, and it is many times helpful to construct such a plane.

105. How a Plane may be Located.—The position of a limited plane may be located in a picture similarly to the method used in descriptive geometry; i.e., (*a*) if the directions of two intersecting lines in it are known; (*b*) if a point and the direction of one line in it are known; (*c*) if the positions of three points in it are known.

In (*a*) the vanishing-trace of the required plane will pass through the vanishing-points of the two lines, and its initial trace through their initial points.

In (*b*) the vanishing-trace will pass through the vanishing-point of the line and also through that of any other line which may be drawn through the given point and intersecting the line. The initial trace will as before pass through the initial points of these lines.

In (*c*) if any two lines are drawn through two of the points and intersecting in the third, or else a line through two of the points and one through the third to intersect the first line anywhere, the conditions will be reduced to those in (*a*).

106. The Line of Greatest Declivity of a Plane.—The line of greatest declivity of a plane may be defined, in a broad sense, as that line which specifies the relative position of the plane, as to angularity, with any other; it is a line therefore which is perpendicular to the line of intersection between it and any other plane. If a plane is passed perpendicular to two intersecting planes it cuts from each lines of greatest declivity with respect to each other. The term has been usually used, however, to define the position of a plane merely with respect to the horizontal coordinate plane.

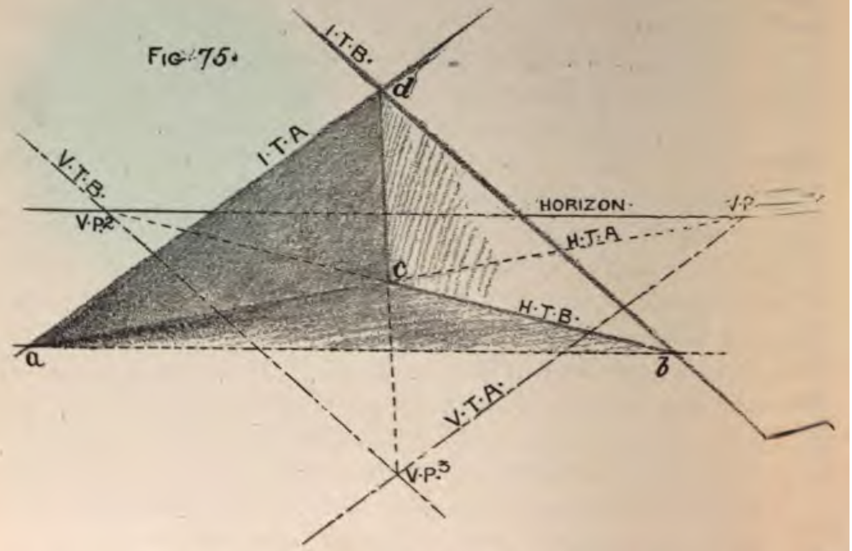
107. The Locus of Planes making a given Angle with the P.P.—If an indefinite number of planes are conceived to be drawn, all maintaining a fixed angle α with the P.P., it is easily seen that they will envelope a conical surface whose elements—the lines of intersection of the various planes—will maintain the angle α with the P.P. The axis of this cone will be perpendicular to the P.P.; hence the angle which the elements make with the axis will be $90° - \alpha$. Now, if lines of direction are conceived to be drawn from the S.P., parallel to these elements, infinite in number, the locus of their intersections with the P.P., i.e., the locus of the vanishing-points of the elements, will be a circle with C.V. as a center. The vanishing-traces of the planes, infinite in number, which are conceived to be drawn, must all be tangent to this circle, that is, the traces themselves will envelope the circle, just as the planes enveloped a cone. The horizontal traces in each case would vanish in the points in which the vanishing-traces cut the horizon.

108. **The Locus of Planes making a given Angle with the Horizontal Plane.**—If an indefinite number of planes are conceived to be drawn, all maintaining a fixed angle β with a horizontal plane, it is easily seen that they will envelope a conical surface whose elements will maintain the angle β with the horizontal plane. The axis of this cone will be parallel to the P.P., and its elements will be at the angle of $90° - \beta$ with this axis. If, as before, lines of direction are conceived to be drawn from the S.P. parallel to these elements, to cut the P.P., it would be found that, since the P.P. is parallel to the axis, the locus of these points of intersection or vanishing-points would be the conic section known as the hyperbola. If the elements of the cone were conceived to be indefinitely extended, two series would be formed, one cutting the P.P. above the other, the two constituting the double nappe of the hyperbola, the upper nappe and the lower nappe. This was also spoken of in section 93. The vanishing-traces of the planes, infinite in number, would be tangent to one or the other of these two nappes or curves; so, as before, the traces will envelop the curves as the planes would envelop the cone. Further, there would be two elements of the enveloped

cone parallel to the P.P.; hence the asymptotes of the hyperbola would be two lines intersecting the horizon in the C.V. at the angle β with the horizon. The vanishing-traces of those two planes coinciding with the asymptotes would be perpendicular to the P.P., and their horizontal traces would be perpendicular to the P.P. also, and hence vanish in C.V.

109. Application to Problems.—The relations of these various traces of a plane to the P.P. and to other planes are best understood by an analysis and solution of a few problems embodying these relations. Since it is desired to obtain a working knowledge of the subject to apply in sketching, particularly in casting shadows, an exact solution will first be given, in conformity with the rules of linear perspective, then

FIG. 75.



an approximate or sketch solution will be discussed, showing how, by estimating convergence of lines, values of angles, and the various perspective conditions, a sufficiently accurate result for all practical purposes may be obtained with much less labor. The first is to make clear the theory involved and the second is that which alone is of practical use to the draftsman.

110. Problem 1.—*Given the initial and vanishing traces of two planes, to find their line of intersection.* See Fig. 75.

Exact Solution.—Let *I.T.A.*, *V.T.A.*, and *I.T.B.*, *V.T.B.*, be the traces of any two planes, and for convenience intersecting within the limits of the drawing. The simplest solution is as follows: Since the line of intersection is common to both planes, one of its points must be at the intersection of the initial traces of the two planes, and the other at the intersection of their vanishing-traces. Hence the line connecting these points will give the picture of the line of intersection. Another solution can be made as follows: Draw any horizontal plane *abc*, to cut the planes *A* and *B*, and whose initial trace will, of course, be a horizontal line *ab*. The line of intersection of the plane *A* with this horizontal plane will be a line drawn from *a* to *V.P.*[1], the point in which the vanishing-trace of the plane *A* cuts the horizon; similarly the line *bV.P.*[2] is the horizontal trace of the plane *B*. Where the two horizontal traces cross each other at *c* must be a point in the line of intersection of the two given planes. The point *d* in which their initial traces intersect will be another point.

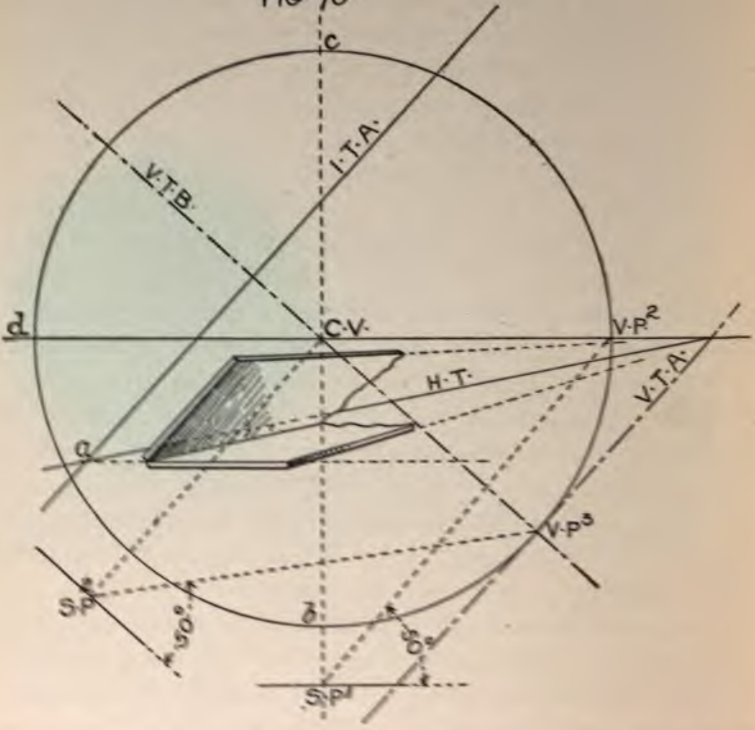
Since no station-point or center of vision is necessary in the solution of this problem, the exact method is as simple as any approximate one could be. If only the initial traces of two planes were given, then any horizontal traces for the planes could be assumed at will, as, in the absence of further data, there would be no limit as to their direction.

From the above may be deduced the solution for a very useful problem, namely, to find the point of intersection of any given line with a given plane. If any plane is passed through the line, its line of intersection with the given plane will cut the given line in the point required. It is often convenient, in the absence of other restrictions, to pass a plane through the given line which is perpendicular to the horizontal plane, hence has a vertical initial trace, and also perpendicular to the given plane, and hence gives as their line of intersection the line of greatest declivity of the given plane with respect to the horizontal plane.

111. Problem 2.—*Given the angle with the P.P., and th initial trace of a plane, to find its horizontal trace.* See Fig 76.

SOLUTION I.—*Exact Method.*—Let the angle with the P.P. be 50°. Following the method described in section 107, if any horizontal line of direction *S.P.*^{1} *V.P.*^{2} is drawn making the required angle with the P.P., it must cut the P.P.

FIG. 76.



in a point on a circle having its center at *C.V.* If *I.T.A.* is the initial trace of the given plane, then the vanishing-trace must be parallel to it and also tangent to the circle *bcd*. Any point *a* in the initial trace can be taken as the initial point of the horizontal trace. The latter must vanish in *V.P.*^{1}, where the vanishing-trace intersects the horizon.

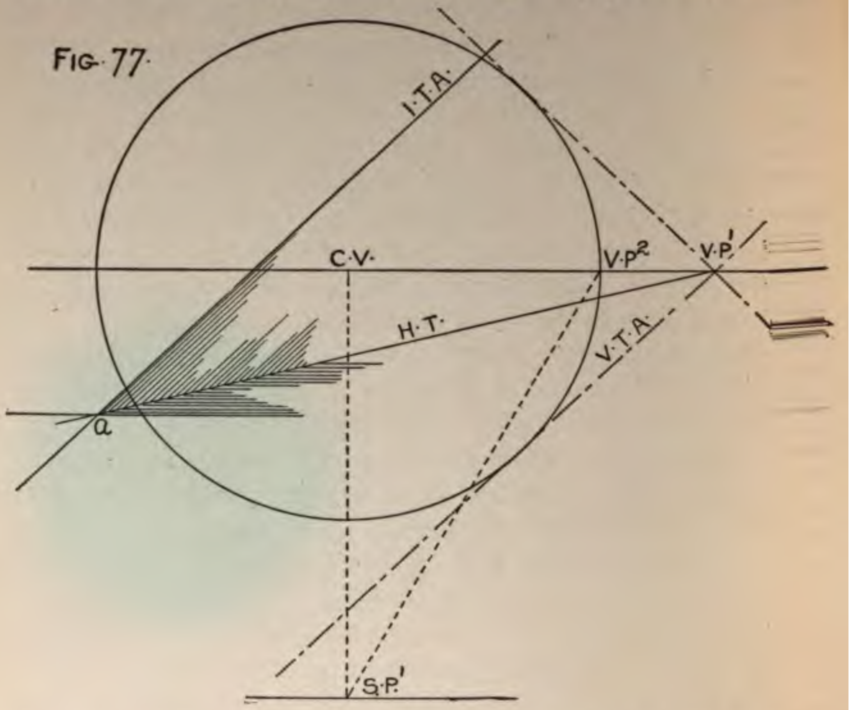
SOLUTION II.—The vanishing-trace, as before, must be found. If an auxiliary plane *B* is conceived to be passed perpendicular to the P.P. and to the given plane, the lines cut out in each of the latter two by the former will be at the angle to each other that the given plane is with the P.P. The vanishing-trace of the auxiliary plane will pass through *C. V.*, which can be taken anywhere, and be perpendicular to the given initial trace. If the *S.P.* is rotated about the vanishing-trace of the auxiliary plane to the position *S.P.*^{2}, then *C. V. V.P.*^{2} will be parallel to the line cut out on the P.P., and *S.P.*^{2} *V.P.*^{3} can be constructed parallel to the line cut out in the given plane, namely, a line at 50° to the P.P. *V.P.*^{3} is the point in which this line of direction cuts the P.P., and will be one point in the vanishing-trace of the given plane, or, in other words, *V.P.*^{3} would be the vanishing-point of all lines of *greatest declivity* of the plane *A* with respect to the P.P. From here on the process is the same as in the first solution.

Approximate Method.—From an inspection of the above demonstration it can be seen that the radius of the circle through *V.P.*^{2} *V.P.*^{3} is determined by the proximity of the station-point to the P.P. The radius will be larger if the station-point is further away, smaller if it is nearer; therefore, if no conditions are fixed as regards the position of the station-point, then almost any line drawn parallel to *I. T. A.* will be the picture of the vanishing-trace, and the problem is exceedingly simple of construction. If, however, the vanishing-trace is made to go through *C. V.*, an impossible case is assumed, namely, that *S.P.*^{1} is coincident with the P.P. at *C. V.*

112. Problem 3.—*Given the horizontal trace and the angle with the P.P., to find the initial and vanishing traces.* See Fig. 77.

Exact Method.—Let the given angle be 60°. This problem differs but little from the preceding one. *V.P.*^{1} will be given or can be obtained by locating a horizon in any desired position with respect to the horizontal trace. If *C. V.* and *S.P.*^{1} are fixed, then, by means of the line of direction *S.P.*^{1} *V.P.*^{2}, the circular locus can be found; this is the same locus as in the preceding problem. The required vanishing-trace must be tangent to this circle and pass through *V.P.*[1]; hence the initial trace can be drawn parallel to it through any point *a* on the

FIG. 77.



horizontal trace. Another vanishing-trace could have been drawn tangent to the circular locus above the horizon.

Approximate Method.—As in Problem 2, almost any line can be drawn through *V.P.*[1] to represent the picture of the required vanishing-trace; as the angle of inclination of the trace to the horizon is made great or small, the *S.P.* will be indicated as a correspondingly greater or lesser distance from the P.P. If the circular locus were of such a radius as to pass through *V.P.*[1], then the vanishing-trace will be a vertical line and the plane it lies in will be perpendicular to the horizontal plane. If the locus were assumed to cut the horizon outside of the point *V.P.*^{1}, with respect to *C.V.*, an impossible case would result.

113. Problem 4.—*Given the initial trace and the angle with the horizontal plane, to find the horizontal and vanishing traces.* See Fig. 78.

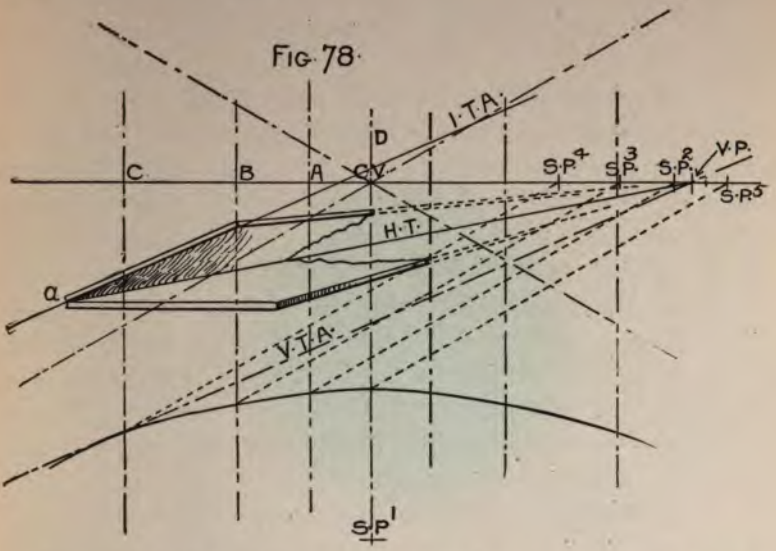


FIG. 78.

Exact Method.—Let the angle with the horizontal plane be 30°. The conditions in this problem are based on those of section 108. The locus of the vanishing-points of the elements of the enveloped cone will be a hyperbola with center at *C.V.* and asymptotes passing through *C.V.* at 30° to the horizon. With *C.V.* and *S.P.*^{1} fixed, the curve or curves of the hyperbola can be sufficiently indicated in position by plotting a few points as follows: In order to draw lines of direction parallel to the elements of the cone, conceive of any number of vertical planes to be passed through the *S.P.* and cutting the P.P. in the lines, *A*, *B*, *C*, etc., one of which, *D*, is perpendicular to the P.P. Rotate the *S.P.* about each of these vanishing-traces in turn to fall on the P.P. at *S.P.*^{2}, *S.P.*^{3}, *S.P.*^{4}, etc. Lines of direction from these rotated positions making 30° with the horizon will cut the respective vanishing-traces *A*, *B*, *C*, *D*, etc., in points which will lie along the curve of the hyperbola. A similar curve could have been drawn above the horizon, thus constituting the upper nappe. Now a line tangent to the curve of the hyperbola, above or below the horizon, and parallel to the initial trace, will be the vanishing-trace of the required plane. A horizontal trace is then easily located.

Approximate Method (see Fig. 79).—From the conditions of the problem it can be seen that, if the observer is assumed to move away from the P.P., the curve of the hyperbola which might be conceived to be constructed from his successive positions would recede from the horizon. Hence, almost any line drawn parallel to the initial trace would constitute the desired vanishing-trace, except such a condition occurs as was discussed in Problem 2; namely, that, as the vanishing-trace is sketched close to the *C.V.*, if that be present as part of the given data, then the *S.P.* approaches coincidence with the P.P. For this reason a line drawn parallel to the initial trace and through the *C.V.* would be an impossible case.

114. Problem 5.—*Given the angle with the horizontal plane and the horizontal trace, to find the initial and vanishing traces.*

Exact Method.—Let the angle with the horizontal plane be 45°, and let *C.V.*, the $S.P.^1$ and the horizontal trace, vanishing in $V.P.^1$, be given as shown in Fig. 80. Assume the initial point of the horizontal trace at any point *a*, then *ab* will be the initial trace of some auxiliary horizontal plane which may be constructed. The angle of the required plane with this horizontal plane is measured (see section 106) between a line lying in the latter and the line of greatest declivity of the required plane with respect to this horizontal plane. Moreover, as before stated, these lines may be conceived to be lines cut out of the two planes by a third auxiliary plane perpendicular to the two. This plane will be a vertical plane in the present problem, and perpendicular perspectively to the hori-

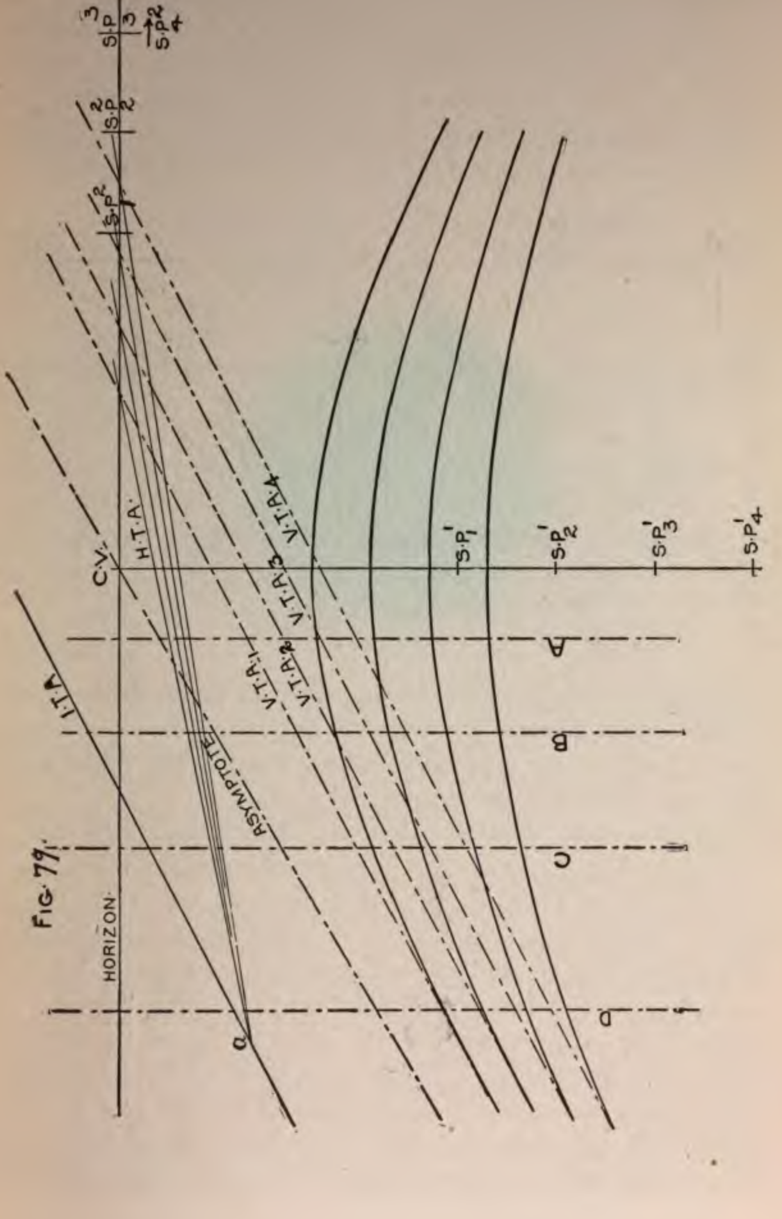
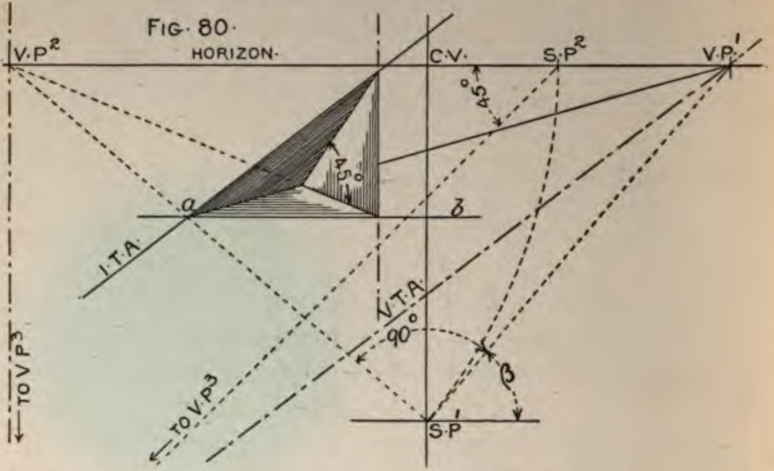

S.P 3
S.P 2
S.P 2/4
S.P 2/1
C.V.
H.T.A.
V.T.A.4
V.T.A.3
V.T.A.2
V.T.A.1
ASYMPTOTE
I.T.A.
HORIZON
S.P'1
S.P'2
S.P'3
S.P'4
A
B
C
D
a

FIG. 79.

zontal trace *aV.P.*[1] Hence its vanishing-trace, which is a vertical line, will go through *V.P.*[2], found by drawing a line of direction from *S.P.*[1] perpendicular to the line of direction *S.P.*[1] *V.P.*[1] A line of greatest declivity of the required plane is next drawn by rotating the *S.P.* about the vanishing-trace of this auxiliary plane into the position *S.P.*[2] and then drawing a line of direction at 45° to the horizon to cut the P.P. in *V.P.*[3]

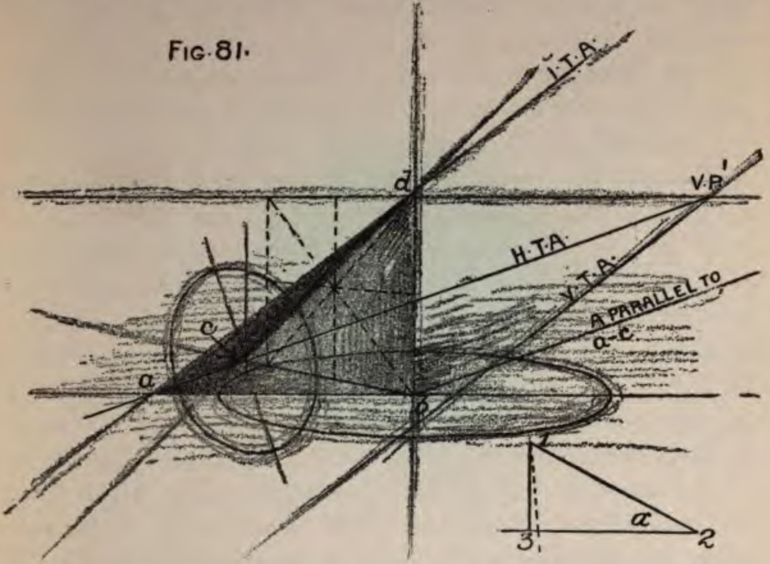
FIG. 80.



Now, since *V.P.*[1] is one vanishing-point in the vanishing-trace of the required plane and *V.P.*[3] is another, a line connecting the two will be the required vanishing-trace. The initial trace will pass through *a* and be parallel to it.

Approximate Method.—Let the horizontal trace be as shown in Fig. 81, and *V.P.*[1] its vanishing-point taken anywhere. Here, as in the preceding method, use can be made of the auxiliary vertical plane; its initial trace can be taken anywhere, as at *bd*, and its horizontal trace afterwards sketched according to the law given in section 72. First, however, in order to do this, a value must be given to the angle with the P.P. of the line *acV.P.*[1] This may be assumed to be any angle within wide limits; after this, a horizontal directed plane can be drawn with its center at *b* and of the ratio of axes which will conform to the assumption made; that is, of such a ratio that were the center of the ellipse at *a*, the line *ac* would cut the perimeter in a point representing the picture of the assumed angle. Next, the horizontal trace of the auxiliary plane is sketched to make the complement of the angle with the P.P. that the horizontal trace of the given plane does. The line of greatest declivity of the required plane will make the picture of 45° with the horizontal trace of the auxiliary plane, and cut the initial trace of that plane in *d*, the point in which the initial trace of the required plane cuts it, since the line is common to the two planes. The last directed plane for estimating this angle can be sketched in a manner similar to the first; its foreshortening can be estimated roughly by that of the auxiliary vertical plane, sketching in it a square as shown in dotted lines in the drawing, or it can be gotten at differently. Let the angle which the auxiliary plane makes with the P.P. be α, as shown in the diagram to one side, then, by drawing any line 1–3 perpendicular to 2–3, the distance 3–2 will be in the ratio to 1–2 that the edges of a square would if constructed in the auxiliary plane standing on one of

FIG. 81.



its edges, neglecting slight convergence to the station-point which is at a finite distance, and the possibility that the square might not be directly in front of the observer. From this square the ratio of axes of the directed plane can be approximately judged. With the angle of 45° constructed at *c*, the point *d* is found and locates, together with the point *a*, the initial trace of the required plane. From this the vanishing-trace can be drawn.

115. Problem 6.—*Given the horizontal trace and initial trace, to find the angles with the horizontal and picture planes.*

Exact Method.—Let the initial trace *I. T. A.* and the horizontal trace be given, as shown in Fig. 82, the latter meeting the former in the point *a*; also let *C. V.* and *S.P.*^{1} be given. After finding the angle α, which the horizontal trace makes with the P.P., pass a plane perpendicular to the given plane and to the horizontal plane as was done in the preceding problem, the vanishing-trace of this auxiliary plane being *V. T. P.* The initial trace of the plane can be assumed to be *bd*, the point *d* being also the initial point of *cd*, a line of greatest declivity of the given plane. By prolonging this line until it meets *V. T. P.* in *V. P.*^{3}, we have a point in the vanishing-trace of the given plane. Now then, if the *S. P.* is rotated about the vanishing-trace of the auxiliary plane into the P.P., obviously the angle β between the horizon and a line of direction from *S. P.*^{2} to *V. P.*^{3} will be the required angle which the given plane makes with the horizontal plane.

In order to find the angle which the given plane makes with the P.P., conceive a plane *V. T. Q.* passed perpendicular to the P.P. and to the given plane *A*. Its vanishing-trace will be perpendicular to the initial and vanishing traces of *A* and pass through the *C. V.* This will cut the P.P. and the given plane each in a line, the lines of greatest declivity with respect to the two, the angle between which it is purposed to measure. Rotate the *S. P.* about the plane *V. T. Q.* into the P.P. to fall at *S. P.*^{3} Then it can be seen that, since the line *S. P.*^{3} *f* is parallel to the P.P., *S. P.*^{3} *e* is a line of direction

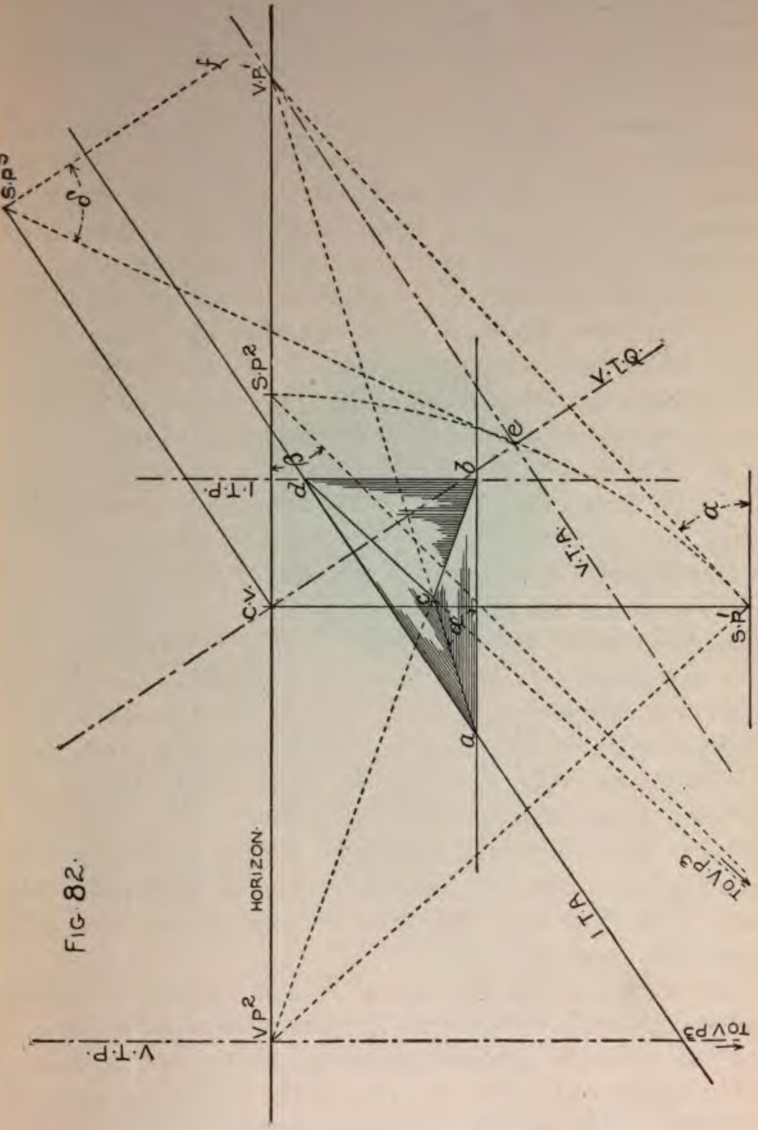


FIG. 82.
S.P³
f
δ
V.P.
S.P²
V.T.Q.
e
β
d
b
I.T.P.
α
V.T.A.
C.V.
c
α
S.R.
a
HORIZON
TO V.P³
I.T.A.
V.P²
V.T.P.
TO V.P³

which is parallel to the line of greatest declivity of the given plane with respect to the P.P. Hence the angle δ between these two lines is the angle which the given plane makes with the P.P.

Approximate Method.—Let the initial trace and the horizontal trace of the plane P be given as in Fig. 83. Draw the vertical auxiliary plane perpendicular to the given plane by estimating the angle of 90° between their horizontal traces as in Problem 5, Fig. 80; *acb* is this angle. By drawing *cd*, the line of greatest declivity of the given plane with respect to the horizontal plane, the angle *dcb* is seen to be the picture of the angle which the given plane makes with the horizontal plane. To estimate the value of this angle a directed plane must be constructed within the auxiliary vertical plane with its center at *d*. The axis of this directed plane is a line parallel to *ac V.P.*[1] Before the proper ratio of axes can be given to this plane, however, some assumption must be made as to the relative values of the angles *bac* and *cba*. After this is decided upon, a square can be sketched in the plane of *bdc* on the assumption made, and an ellipse inscribed in this, which will give approximately the ratio of axes desired. Or another method of arriving at the ratio of axes would be to make a diagram similar to that made in Problem 5, shown to one side, the estimated angle α between the horizontal trace of the auxiliary plane and the P.P. being shown its actual value. This is only a rough approximation, though accurate enough for all practical purposes, assuming, of course, that the directed plane of the diagram is on a level with the eye and directly in front of the observer.

To ascertain approximately the angle which the plane makes with the P.P., draw the vanishing-trace of the plane Q as in the preceding problem, perpendicular to the P.P. and to the vanishing-trace of the given plane, the $C.V.$ being assumed anywhere. The point e, as before, in which this trace cuts the vanishing-trace of the given plane, will be the vanishing-point of the line of greatest declivity of the given plane with

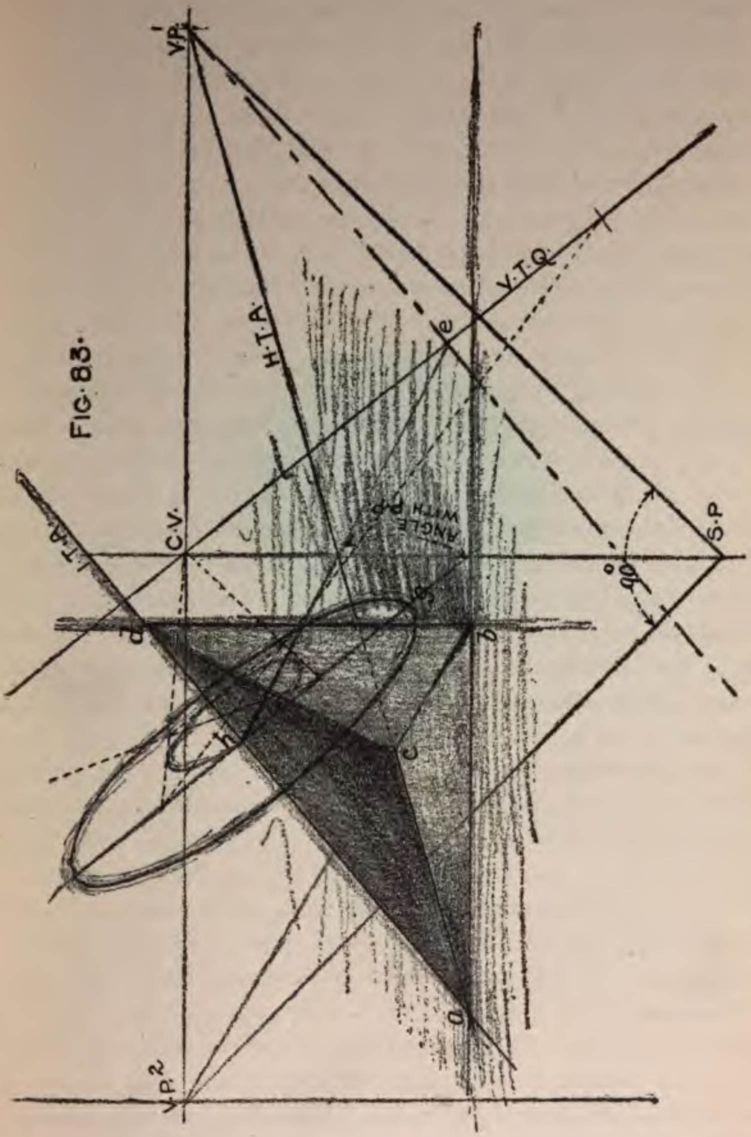


FIG. 83.

respect to the P.P. The angle between it and a line meeting it and lying in the P.P. and parallel to *V.T.Q.* remains to be estimated, namely, the angle *ehg*, *h* being a point taken anywhere in the trace of *A*. To do this a directed plane of rotation can be drawn with its center at *h* and its plane parallel to that of *Q*. The proper ratio of axes for this directed plane cannot be found until the position of the *S.P.* is located. This can be done by a slight inspection of the conditions of the problem. *S.P.* must be opposite to *C.V.* as assumed, and at such a distance from the horizon that lines drawn from it to *V.P.*[1] and *V.P.*[2] respectively will make 90° with each other. This distance can be approximated by eye closely enough for all practical purposes. The ratio of axes of the desired directed plane can be obtained by drawing the circumscribing square whose initial trace is parallel to *V.T.Q.* and whose diagonals vanish at a distance from *C.V.* equal to the distance the observer is from the P.P. (see dotted construction). This square need not really be drawn entire in order to obtain the ratio desired, because a slight difference in ratio of the axes makes an even smaller relative difference in the value of the angle it is desired to estimate by means of the directed plane. After a plane similar to this is sketched with *h* as a center, the angle *ehg* can be judged fairly closely.

116. Problem 7.—*Given the angle which a plane makes with both the horizontal plane and the picture-plane, to find its possible vanishing-traces.*

Exact Method.—Let the angle with the P.P. be 70°, and that with the horizontal plane be 30°. (See Fig. 84.) In section 107, the vanishing-traces of planes making a given angle with the P.P. were shown to intersect each other and envelop a circle whose center was at *C.V.* Also, in section 108, it was shown that the locus of the vanishing-traces of planes making a given angle with the horizontal plane would intersect each other and envelop a hyperbola, whose asymptotes, passing through the *C.V.*, would be inclined to the horizon at the angle which the required planes made with the horizontal plane.

It is not difficult to draw the deduction from these facts that, in order to fulfil the conditions of given angles with both planes, the required planes must be tangent to two cones whose axes are respectively parallel and perpendicular to the P.P. Hence, to find their vanishing-traces, two cones are conceived to be constructed with their apexes at the *S.P.*, and in the one, lines of direction or elements of the cone drawn at an angle of 70° to the P.P., and in the other, lines of direction drawn at an angle of 30° to H.P. Fig. 84 shows the construction in detail. The required traces are tangent to both the hyperbola and the circle. It is seen that four lines can be drawn tangent to these curves, hence there can be four planes fulfilling these conditions.

There are two facts of interest and importance to note in connection with this problem: (1) If the angle which the given plane makes with one of the reference planes is the complement of the angle it makes with the other, then the vanishing-traces will all reduce to two, both parallel with the horizon, one above and one below it. (2) The sum of the given angles with the P.P. and H.P. must be equal to or greater than 90°, otherwise the circle would intersect the hyperbola and the case would be an impossible one.

For this problem no approximate sketch method can be used to fulfil definite conditions, because the fixing of one of them limits immediately all the rest. For example, consider as placed the circular locus of the vanishing-points of the traces of planes at a given angle to the P.P.; for this given angle the station-point is also fixed; this in turn fixes the position of the other locus, the hyperbolic one. Or again, the conditions are limited because each trace has but one position and one direction for a certain position of the station-point and the given angles; for any change of distance of the station-point, a new trace will be found parallel to the first; for any change in values of the angles and a fixed station-point, the trace wi[illegible] have a new direction.

117. An Inverse Process Illustrated.—Before leaving th[illegible] subject of the horizontal and vertical traces of a plane and it[illegible] angle with either the horizontal or picture plane or both, [illegible] will be instructive to take a problem in which a limited pla[illegible] is given and, by means of it, ascertain the probable perspectiv[illegible] conditions together with the angles which the plane mak[illegible] with both reference planes.

Let *abcd*, Fig. 85, be a sketch of a square plane supposedl[illegible] standing on an edge and inclined obliquely with respect t[illegible] both the horizontal plane and the picture-plane. By prolon[illegible]ing its horizontal edges, $V.P.^1$ may be found and the horizo[illegible] located; then assume $C. V.$ as shown, the observer being o[illegible]posite to it when the square was drawn;* from the appare[illegible] foreshortening of the square make a further assumption as t[illegible] the probable value of the angle α which the lower edge mak[illegible] with the picture-plane; the angle might also be estimated b[illegible] use of the directed plane of the line *ab*, plotting three othe[illegible] points in the ellipse, symmetrical like *b* with respect to *a*. Th[illegible] estimated value of the actual angle struck off at $V.P.^1$ serve[illegible] to locate $S.P.^1$ To measure the angle of the square with th[illegible] horizontal plane, it is convenient to pass a vertical plane throug[illegible] the edge *ac*; this vertical plane will intersect the horizonta[illegible] plane in the line $a\ V.P.^2$, which line, being at 90° to *ab*, i[illegible] found by constructing such an angle at $S.P.^1$ with respect t[illegible] $S.P.^1\ V.F.^1$, giving $V.P.^2$ as its vanishing-point and $V.T.P$[illegible] as the vanishing-trace of its plane. To prolong the edges a[illegible] and *bd* till they meet, as they should if the sketch is correct[illegible] on $V.T.P.$ at $V.P.^3$, gives the means for measuring the angl[illegible] and checking the assumptions made. For, if the $S.P.^1$ i[illegible] rotated about $V.T.P.$ to $S.P.^2$, the angle β which a line o[illegible]

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* A person in sketching such a figure would probably take a position with reference to the square and the P. P. so that *C. V.* would lie somewhere between two vertical lines through *c* and *b*.

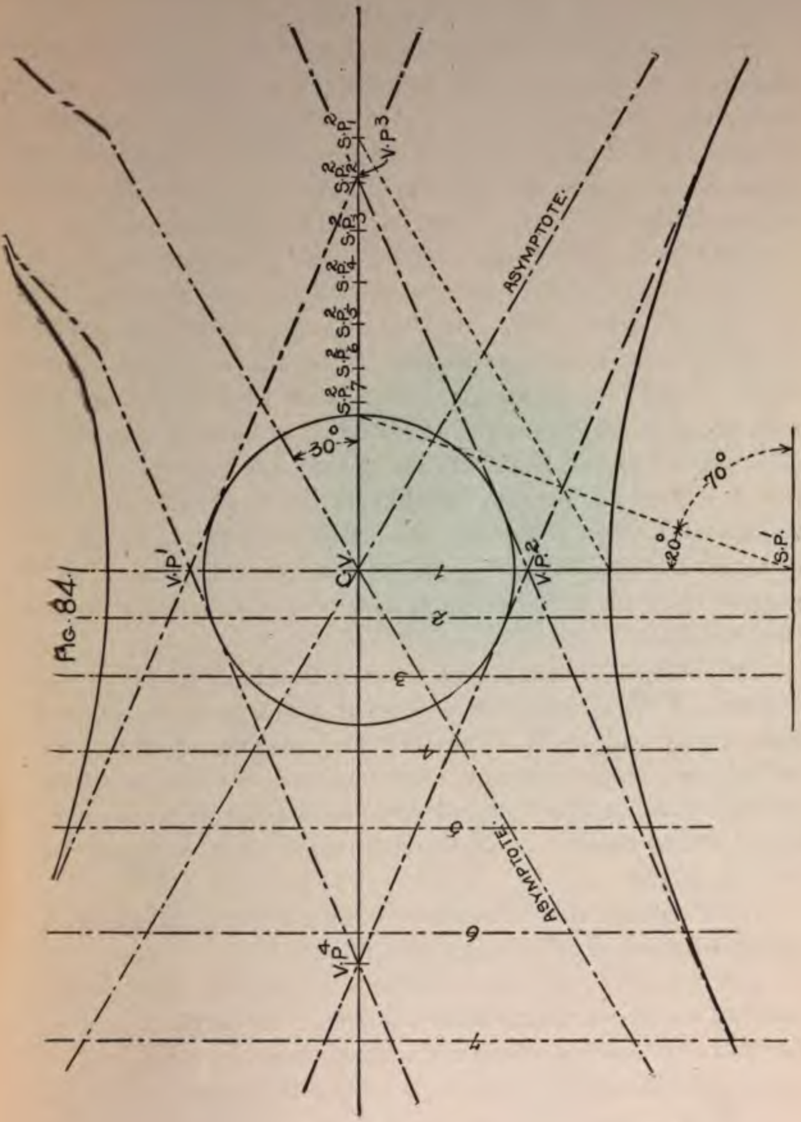
V.P.1
C.V.
V.P.2
S.P.1
30°
70°
20°
ASYMPTOTE.
ASYMPTOTE.
V.P.3
V.P.4
S.P.2 1
S.P.2 2
S.P.2 3
S.P.2 4
S.P.2 5
S.P.2 6
S.P.2 7
1
2
3
4
5
6
7

FIG. 84.

direction from $S.P.^2$ to $V.P.^2$ makes with the horizon will be the measure of the angle desired.

To ascertain the angle with the P.P. a somewhat longer process is necessary. The initial trace of the plane of the square must first be found. This is done by passing an auxiliary vertical plane through the further edge *hd* of the square; this plane it can easily be seen, by following the construction, intersects the P.P. in the line *fg*. The intersection of the line *hd* with this initial trace in the point *f*, gives one point in the initial trace of the plane of the square; *a* is another, and the two locate the trace. Now since $V.P.^2$ is one point in the vanishing-trace of the square, this can be drawn parallel to the initial trace. The same process is now employed that was used in problem 6, page 150, namely, a plane is conceived to be passed perpendicular to the plane of the square and the P.P., $V.T.Q.$ being its vanishing-trace, and the observer rotated about this trace into the position $S.P.^3$. A line of direction $S.P.^3$ $V.P.^4$ will give, by its angle with $V.T.Q.$ or a parallel to it, the angle δ which the plane of the square makes with the P.P.

118. Miscellaneous Theorems and Problems in Lines and Planes.—Following this are collected a few problems in lines and planes which it is intended the student should solve entirely by sketch methods, unless otherwise stated, working with the initial and horizontal traces only as given or required. Before doing these a certain few geometrical facts should be fixed in mind.

(1) If two parallel planes are cut by a third plane, the lines of intersection cut out in each will be parallel.

(2) If a line is perpendicular to a plane, the right projection* of the line on any other plane will be perpendicular to the line of intersection between the two planes.

(3) It is also well to recall to mind how the angle between

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* By right projection is meant one in which the projecting plane is perpendicular to the plane on which the line is projected.

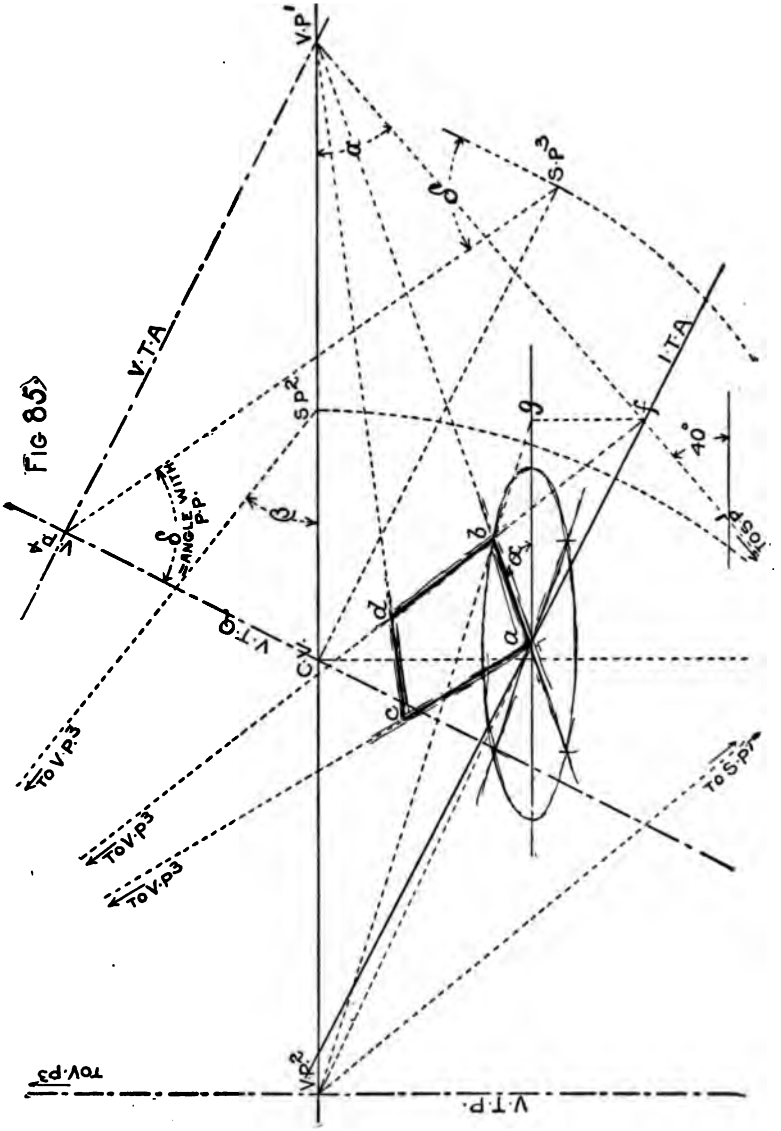


FIG 85.

V.P¹
V.T.A
S.P³
SP²
C.V.
V.T.Q
I.T.A
VP²
V.T.P.
To V.P.3
To V.P3
To V.P3
TO V.P3
To S.P1
To S.P
α
β
δ
δ
= ANGLE WITH P.P.
40°
a
b
c
d
f
g
α

any two planes is measured. Perspectively this angle may be foreshortened, and if the directed plane of a rotating line is used by which to estimate the angle, the axis of the directed plane is the line of intersection of the two planes. It is therefore necessary either to pass a plane perpendicular to the two planes or else to *estimate* the degree of foreshortening of the directed plane so as to ascribe the proper value of axes for the ellipse.

PROBLEM 1.—To project a line on a plane.

Solution.—In the absence of other limiting conditions any plane passed through the line and cutting the given plane will suffice to project it. The line will be projected in the line of intersection of the two planes. Hence find the initial and horizontal traces of the given line, which may be done by passing a vertical plane through it. Through its traces draw parallel lines to cut the given plane in points which represent the projection. But the usual construction desired is the right projection, in which the auxiliary plane passed through the line should be made perpendicular to the given plane, and the projecting lines from the extremities of the given line should also be drawn perpendicular to it.

PROBLEM 2.—Given a plane *P*, and a limited line in it, to draw other lines in the same plane, intersecting this line at angles of 30°, 45°, and 60°.

Solution.—After making some assumption as to the obliquity of the plane to the horizontal and picture planes, see section 117, or, ascertaining these by analysis, a directed plane can be drawn with any point of the given line as a center, and lying in the plane *P*, after which the required angles can be estimated.

PROBLEM 3.—Through a given line to pass a plane perpendicular to a given plane. Exact method.

Solution.—From any point of the given line draw a perpendicular to cut the given plane; then find the horizontal and initial traces of these two lines: they will be points in the horizontal and initial traces of the desired plane.

119. General Conditions for Tangency.—If a plane is tangent to a surface, it will depend upon the shape of that surface what the nature of the tangency is, that is, whether it will be tangent throughout the plane surface, along an element of the surface or at a point, etc. The general conditions of a tangent plane may be stated as follows:

A plane is tangent to a surface when it has at least one point in common with the surface through which, if any other cutting plane be passed, the line of intersection of the two planes will be tangent to the curve cut from the surface. The tangent plane will therefore contain all lines tangent to the surface at the point of contact. Hence, to draw a plane tangent to any surface, the general rule would be to draw any two tangents or pass any two cutting planes, and, by means of tangents to the curves cut from the surfaces, obtain the traces of the required plane.

A plane which is tangent to a surface of revolution, i.e., a sphere, circular cylinder, cone, etc., will be perpendicular to a plane passed through the point or element of contact and containing the axis of the solid.

The perpendicular plane spoken of which contains the axis of the solid is generally known as a *meridian plane*, and the line on the surface cut out by it is known as the *meridian line or curve*.

Surfaces of revolution will be the kind of surfaces referred to, and it may be as well to state that, unless otherwise specified, a cylinder or cone when mentioned will mean a right circular cylinder and cone respectively. A right circular cylinder is one whose elements are perpendicular to the circular base, and a right circular cone is one whose base is a circle and its axis is perpendicular to the base at its center.

A plane tangent to a sphere will touch it in only one point, and of course will be tangent to any curve of section of the sphere through that point.

If a plane is tangent to a cylinder, it will be tangent throughout an element of the cylinder, that is, throughout a line on its surface which is parallel to the axis of the cylinder; it would, from what was said above, be tangent to the curves of the bases of the cylinder, or to the curve of any section of the cylinder taken transversely of its axis. Further, if such second transverse plane is drawn to cut the cylinder and the tangent plane, then the line of intersection of the two planes would be tangent to the curve of the section of the cylinder formed thereby.

If a plane is tangent to a cone, it will be tangent throughout an element of the cone, and the same conditions hold true with regard to this plane and the base or bases of the cone as were spoken of in connection with the cylinder in the preceding paragraph.

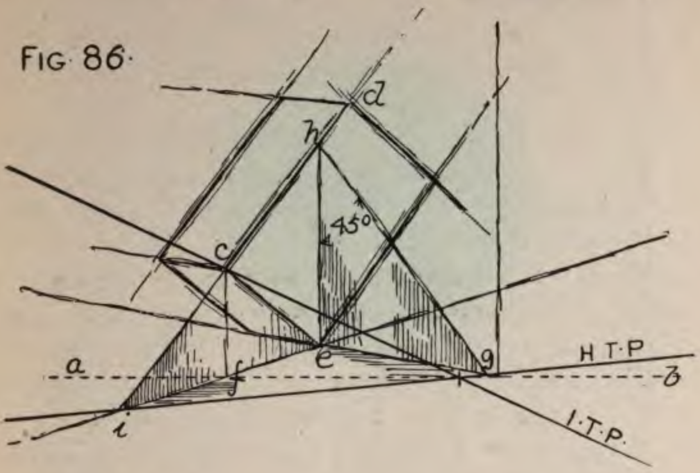
Sometimes it is necessary to consider a plane which touches a rectangular or prismatic solid along one edge; this may also be regarded as a condition of tangency, in which the edge can be called an element of the solid.

120. A Plane Tangent to a Solid of Plane Faces.—The last case of tangency is perhaps the simplest one to comprehend and sketch perspectively, hence we will consider an example of this kind. In point of fact, though it may be most logical to define the various forms of tangency in the order above given, it is probably better to work problems in illustration of them in the reverse order.

If a solid of the prismatic class be placed with its axis vertical, and hence its elements parallel to the P.P., the problem of drawing a plane to contain one of these elements at any angle to the P.P., or to the faces of the solid, presents no perspective difficulties. The tangent plane will be vertical and its vanishing-trace and horizontal trace can be found in the manner previously described. Consider, then, the following case: A block stands as in Fig. 86. Let it be required to pass a plane through the edge *cd* and making an angle of 45° with the horizontal plane.

Sketch the horizontal trace of the plane of *ecd* to make the complement of the angle with the P.P. which the short edges of the block seem to make: the horizontal trace of the edge *cd* will be at *i*. Assuming *c* to be the initial point, drop a vertical line from it, obtaining the point *f*; from it *ab* can be located, the initial trace of a horizontal plane passed through the lower edge of the block. Now, since the required plane is to contain the edge *cd*, its horizontal trace will contain the horizontal trace of this line, namely, the point *i*. Next, pass an auxiliary vertical plane perpendicular to the face *ecd*, its horizontal trace lying in *ge*. The angle *egh* can then be estimated as 45°, giving *g*, another point in the horizontal trace of the required plane. Hence *ig* is that trace. Where it crosses *ab* will be the initial point of the horizontal trace and also one point in its initial trace; *c* is another point in it, hence *I. T. P.* is the required initial trace. A portion of the construction, it is seen, lies in front of the P.P. The vanishing-trace would not be necessary unless exact construction were required.

FIG. 86.



121. A Plane Tangent to a Cone of Revolution.—Let it be required to pass a plane tangent to a cone; a very simple case is that shown in Fig. 87. Suppose the element of contact of the tangent plane to be given as *o*1, the cone standing on its base on the horizontal plane, and the horizontal and initial traces of the tangent plane to be obtained.

Assume the curve of the base to touch the P.P.; then *ab* is the initial trace of the horizontal plane of the base. The horizontal trace of the required plane will be tangent to the curve of the base of the cone, and by sketching such a tangent, 1*c*, this trace is located; *c* will also be a point in the initial trace of the plane. To get another point in that trace, it is most desirable to prolong the element *o*1 until it cuts the P.P., as it is a line lying in the tangent plane; to do this pass a vertical auxiliary plane *Q* through the element; this plane will pass through the axis of the cone and the center of the base and is easily located. *d* is therefore found to be another point in the initial trace of the required plane. If it were also desired to know the angle which the plane *P* would make with the P.P., it would be necessary to make an assumption as to the location of *C.V.* and, from that, obtain the *S.P.*[3].

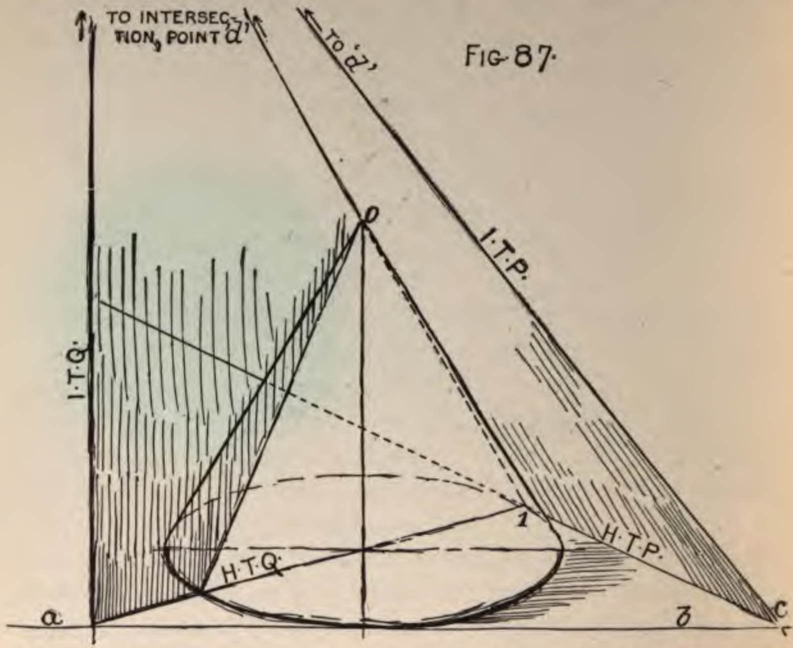
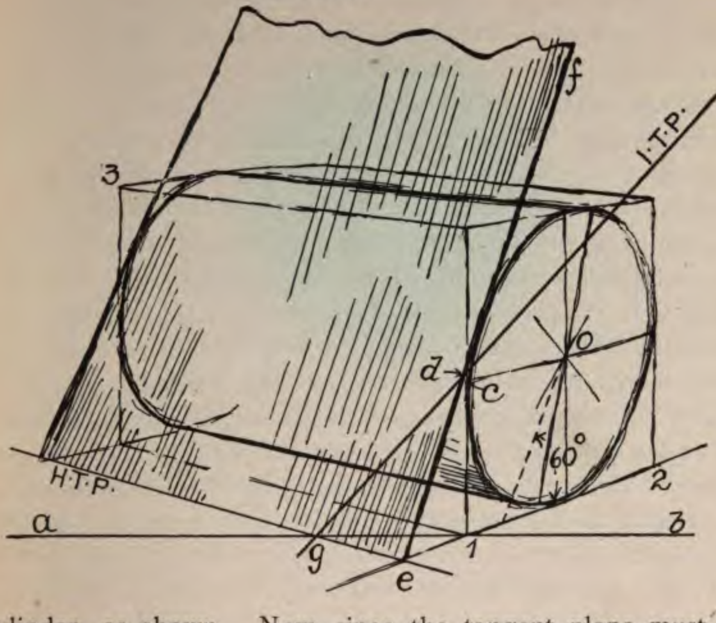


FIG-87·

122. To Pass a Plane Tangent to a Horizontal Cylinder.—Let a sketch of the cylinder be given as shown in Fig. 88, and let the tangent plane make an angle of 60° with the horizontal plane and directed away from the observer.

Assume that the cylinder just touches the P.P. at the point *c*; then *ab*, the initial trace of the horizontal plane, can easily be located; it is very convenient for the purposes of construction to sketch a rectangular solid, 1, 2, 3, surrounding the cylinder, as shown. Now, since the tangent plane must be parallel to the axis of the cylinder, a vertical auxiliary plane, drawn perpendicular to that axis, will cut out a line in the tangent plane which is tangent to the cylinder, and also a horizontal line from the horizontal plane; the angle between these two lines will be the angle which the tangent plane makes with the horizontal plane. The plane of the near base of the cylinder serves the purpose, therefore the direction of a line which is tangent to the cylinder, and makes an angle of 60°

FIG. 88.



with the horizontal plane, is shown by the dotted line through *o*, and the actual tangent to the cylinder in this plane is shown by the line *fe*, sketched parallel to it, *e* being the point at which it touches the horizontal plane. The horizontal trace of the required tangent plane will go through *e* and be parallel to the axis of the cylinder; *g* is one point in the initial trace of the required plane and *d* is another, found by the intersection of *fe* with the P.P. in the line 1*c*.

123. To Pass a Plane Tangent to a Sphere.—Probably the simplest case of tangency to a sphere is when the point of tangency is given. Then any line drawn through this point perpendicular to the radius of the sphere at the point will lie in the tangent plane. The method used is to pass an auxiliary plane through the point and the center of the sphere. This plane will cut a circle out of the sphere, and will be perpendicular to the required tangent plane. The horizontal trace of a tangent to the sphere within this auxiliary plane will be one point in the horizontal trace of the required plane. Any other auxiliary plane may be drawn similarly through the same points and by means of a similar tangent line another point in the horizontal trace of the required plane found. The problem of tangency to a sphere does not occur as often as the other cases described, and it is moreover difficult to sketch graphically on account of the ellipses showing the cross-sections.

124. Planes Intersecting Solids.—The general method of finding the point of intersection of a given line with any surface is to pass any auxiliary plane through the line to cut the surface. Where the given line prolonged cuts the curve of intersection of the auxiliary plane and the surface will be the point required. Practically it is best to choose that auxiliary plane which will cut the simplest kind of curve from the surface, straight lines if possible.

If a plane intersects a solid, the lines of intersection form a plane surface, differing in character according to the shape of the solid and the direction in which the plane cuts it. The phenomena of cast shadows form problems in intersecting planes, and it is desirable to have some insight into the character of certain sections of well-known solids.

A plane cutting a sphere must always cut it in a line which is a circle and, in general, perspectively an ellipse.

A plane may cut a cylinder in one of several different ways. If it is a circular cylinder with circular bases, then, if the cutting plane is parallel to the axis, it will cut a rectangle from the cylinder. If the plane is perpendicular to the axis, the shape of the section is a circle. In any other direction it is in whole or in part an ellipse.

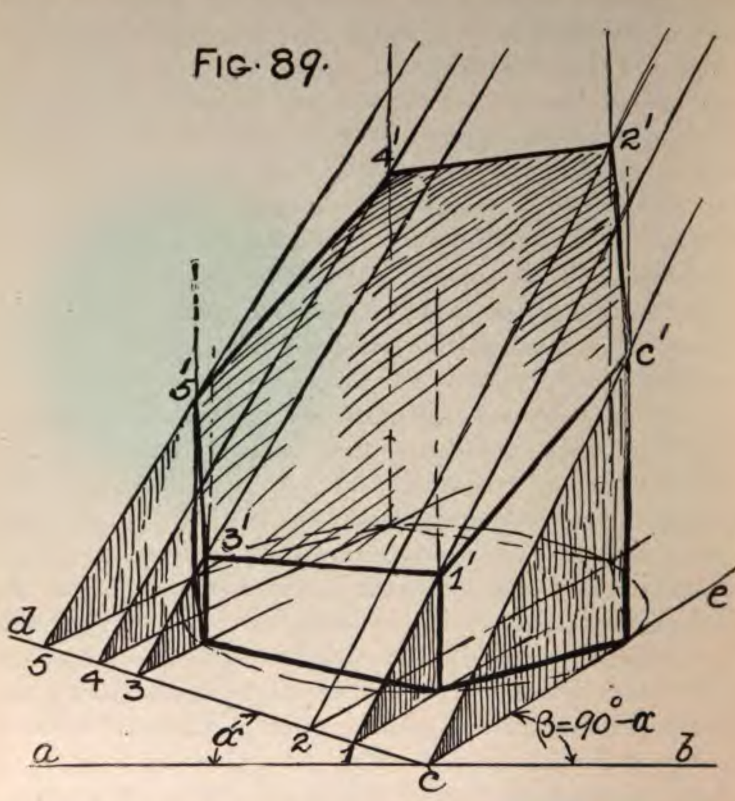
A plane may cut a cone also in one of several different ways. If the cone is a cone of revolution with a circular base, and the cutting plane is passed through the axis, the section will be an isosceles triangle. If the plane is perpendicular to the axis, the shape of the section will be a circle. If the plane is parallel to the axis, we have the hyperbola as the curve; if parallel to an element, then the parabola. If the cutting plane has any other direction, the curve will be in whole or in part an ellipse.

In order to draw any particular and given cutting plane of a solid and deduce the nature of the cross-section, one general plan of procedure is usually adopted, and that is to pass auxiliary cutting planes which will cut straight lines out of the given plane and the most simple lines out of the solid, straight lines if possible. The intersections respectively of the two sets of lines give points in the curve of the section. With a little practice in problems in cutting planes it becomes possible to predict, with a sufficient degree of accuracy, the perspective appearance of the cutting plane. A few problems in intersections are given below.

125. A Plane Cutting a Hexagonal Prism Obliquely.—Given the hexagonal prism standing on its hexagonal base as shown in Fig. 89, also *cd* as the horizontal trace of a cutting plane which is to be at 40° to the horizontal plane, find the shape of the section.

Let any point *c* be the initial point of the horizontal trace, which makes an angle α with the P.P.; planes passed perpendicular to this plane will have their horizontal traces at the angle $\beta = 90° - \alpha$ with the P.P., or parallel perspectively to the line *ce*. According, then, to section 124, pass planes which will be perpendicular to the given plane and which will cut the prism in straight lines. It is most convenient in this case to pass them through the several vertical edges of the prism.

FIG. 89.



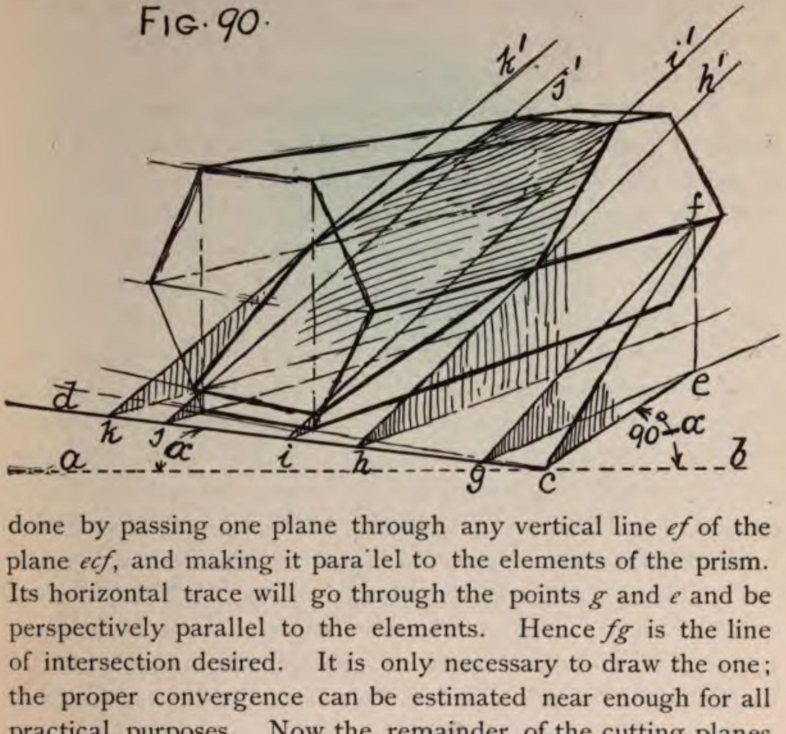
c, 1, 2, 3, 4, 5, are the points in which these several planes cut the horizontal trace of the given plane, and *c*', 1', 2', 3', 4', 5', are the points in which the lines cut out in each plane meet those respectively cut out of the prism.

Suppose the hexagonal prism had the position shown in

Fig. 90, and for variety let the cutting plane make an angle of 30° with the horizontal plane.

Let *cd* be the horizontal trace of this cutting plane, and *ce* that of a vertical plane which may be passed perpendicular to it. Now, as before, it is desirable to cut straight lines out of the prism and the plane, and the most convenient way to do this is to pass vertical planes through the edges of the prism. The direction of the lines of intersection of these planes with the given plane must first be ascertained, however. This is done by passing one plane through any vertical line *ef* of the plane *ecf*, and making it parallel to the elements of the prism. Its horizontal trace will go through the points *g* and *e* and be perspectively parallel to the elements. Hence *fg* is the line of intersection desired. It is only necessary to draw the one; the proper convergence can be estimated near enough for all practical purposes. Now the remainder of the cutting planes will have their horizontal traces parallel perspectively, and their traces with the given plane also parallel, so that the points of the section can be obtained; *hh'*, *ii'*, *jj'*, and *kk'* are the several lines of intersection between the planes passed through the edges of the prism and the given plane *dcf*. Where these

FIG. 90.

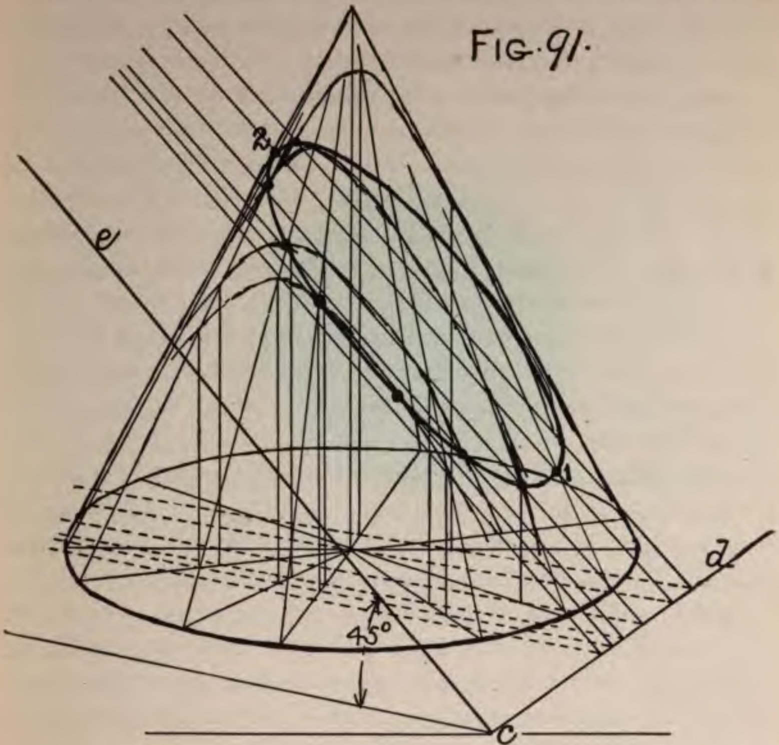


traces intersect the elements of the prism respectively, will be points in the plane of the section desired.

126. A Plane Cutting a Cylinder Obliquely.—The method of ascertaining the shape of the section in the case of a plane cutting a cylinder is the same as that given for the hexagonal prism; only a few words of further explanation are necessary. With the cylinder standing on its circular base, the auxiliary cutting planes would be passed, each to cut two elements from the cylinder, except those two planes which would be drawn tangent to the cylinder. If the cylinder were lying on one of its elements, the process would be similar and according to the second solution of the preceding problem.

127. Given a Circular Cone of Revolution Standing on its Base, to Pass an Oblique Plane Cutting it and at an Angle of 45° to the H.P.—Let the cone be as shown in Fig. 91; it would be possible to pass planes as described in section 124, so as to cut straight lines out of the cone, namely, by causing the planes to go through the apex of the cone; but this is a case in which such a procedure would be more difficult of solution by sketching than to pass planes perpendicular to the given plane and to the plane of the base of the cone, which in this case is the horizontal plane. In the first-mentioned method it would be necessary to find the line of intersection of each plane with the given plane by joining the points of intersection, severally, of their initial traces with the points of intersection of their horizontal traces as described in section 110. The traces, however, would involve considerable constructive work, and this would be more difficult to execute correctly than if the second process were followed, which is largely approximate. This second process is the one which is shown in the figure. *cde* is the cutting plane at 45° to the horizontal plane, *c* is the initial point of its horizontal trace, and *ce* shows the slope of the plane with respect to the horizontal plane, that is, it is a line of greatest declivity with respect to that plane. Vertical auxiliary planes are then passed whose lines of intersection with the given plane will be parallel to *ce* perspectively. We know that the section of the cone will be an ellipse from the slope of the cutting plane, hence one auxiliary plane if passed through the axis of the cone, cutting straight lines from it, will give the points 1 and 2 in this section. Any other of the auxiliary planes will cut hyperbolas from the cone as shown in the figure, the construction being sufficiently clear that but little further explanation is necessary. To get the points of the curves a second set of auxiliary planes are constructed passing through the axis of the cone in each case, cutting elements from the cone; the intersection of these elements with the lines of intersection of the planes give points in the curves. By inspection it can be decided what curves are necessary, and also that only

FIG. 91.



a small portion of each is needed. Three points are sufficient in each, and but two or three hyperbolic sections are necessary, so that 4 or 6 points in the curve of the elliptical section may be obtained.

128. Given a Cone Standing on its Base to Draw a Horizontal Plane Cutting the Cone and to Show the Section.—This is a good problem for practice; therefore only a few words of direction will be given and no figure. If any horizontal line be taken above the base as the initial trace of the horizontal cutting plane, then planes passed through the axis of the cone will cut horizontal traces out of the cutting plane parallel to those cut from the base of the cone. These traces will in the case of each horizontal plane all intersect the axis of the cone in a point. The convergence, then, of any two traces, one above the other, can be estimated and this point found. The planes will cut elements from the cone, and the points of intersection of these elements severally, with the horizontal traces, will give points in the curve of the ellipse which represents the shape of the section.

129. Miscellaneous Problems in Intersecting Planes.—Following are some problems including those in planes tangent to and cutting solids, which as before the student is expected to execute by sketch methods unless otherwise stated.

Problem 1. Draw a square plane perpendicular to the P.P. and making an angle of 45° with the horizontal plane and standing on one of its edges. Consider the square as rotated about its initial trace; find the locus of the vanishing-points of its diagonals. By exact method.

Problem 2. Given a plane making an angle of 20° with the horizontal plane and 70° with the P.P., to find its vanishing-trace. Exact method.

Problem 3. Assuming the eye on a line with the top of a sphere, sketch a circumscribing tetrahedron of the sphere, proportioned as follows: The sides of the tetrahedron make angles of 60° with its equilateral triangular base, which latter is horizontal.

Hint.—Draw the lines of greatest declivity of the faces of the tetrahedron tangent to the sphere; they are the altitude lines of the faces and are at 90° to the base.

Problem 4. Sketch a circular cylinder lying on one of its elements on the horizontal plane, its axis at about 60° to the P.P. Sketch a plane cutting it in the middle and perpendicularly to the axis. Sketch another plane tangent to the cylinder and at 60° to the horizontal plane. Show the lines of intersection of the planes and their initial and horizontal traces.

Problem 5. Sketch a square prism 3 units long by 1 unit square, standing on a corner, its axis inclined at 45° to the horizontal plane, and the faces of the prism inclined likewise, angles optional, the axis also to be so placed that a vertical plane passed through it would cut the horizontal plane in a line at 60° to the P.P. Pass a plane to cut the long edges of the prism and oblique to its rectangular faces. Find its horizontal and initial traces.

Problem 6. Given a circular cone lying on the horizontal plane on one of its elements, and taking any element as one of contact of a tangent plane, find the horizontal and initial traces of it.

Problem 7. Sketch any cone lying on an element, the element inclined to the P.P. Sketch the horizontal and initial traces of a plane tangent to the cone and passing through a given point outside of it.

Hint.—Draw a line through the apex of the cone and the point; from where this line intersects the plane of the base of the cone draw a tangent to the base. The required plane contains these two lines.

Problem 8. Sketch an oblique cone with a circular base on which it is standing, the axis at [illegible] to the base. Incline the axis with respect to the P.P. Cut the cone with an auxiliary vertical plane perpendicular to the axial plane* of the cone, and show the shape of the section.

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* The "axial plane" is that plane with respect to which the cone is symmetrical.

Problem 9. Sketch a circular cylinder standing on its base. Cut it by an oblique plane at 60° to the H.P., and whose horizontal trace is also oblique to the P.P., and show the shape of the section.

Problem 10. Sketch an oblique cylinder, with circular base, in a horizontal plane, axis inclined to the H.P. and the P.P. Cut it by any oblique plane, and show the shape of the section.

CHAPTER X.

THE PHENOMENA OF SHADOWS AND THEIR PERSPECTIVE TREATMENT.

130. Differences in Light and Shade Values Make Objects Visible.—In what has preceded the perspective has been kept to the limits of lines and planes and their possible appearance, upon the supposition that most objects have intersecting surfaces which can be represented by lines in a drawing. This is distinctly an arbitrary treatment because the separation between surfaces, using this illustration as a simple case, is not a line, but an edge of more or less definiteness according to the smoothness of the surfaces. To draw a line on paper calling it this edge is but a convention which by mutual agreement is thus interpreted. The truth, however, is that objects are visible because the light striking them in a certain way is reflected differently to the eye from different parts of the object and hence tells the story of the form. It tells the story through the phenomena of apparent color and of differing degrees of light, often arbitrarily, and somewhat erroneously termed light and shadow. It is not the intention to discuss color effects, but the phenomena of light and shadow, being capable of certain perspective treatment, will be considered in this chapter.

131. Difficulty of Empiricising the Knowledge of Shadow Effects.—In stepping thus from the realm of the conventional into that of the actual and real, it must be confessed at the outset that there are insurmountable difficulties in the way of establishing definite principles concerning shadow effects. The complete representation of form in light and shade is distinctly an artistic accomplishment and can only be acquired, if at all, not through a study of linear perspective, but by a careful and thorough study of form, rendering it in light and shade in some pliable medium, and supplementing it by a thoughtful observation of light effects in general.

It has just been said that all form is visible because of different effects of light, and that these differing effects are caused by the differences in degree with which the light is reflected from the object to the eye. Differences thus reflected depend upon a variety of causes, chief of which are, the direction of the light, the point of view from which the object is seen, the character of the surfaces receiving the light, that is, the quality, smoothness, etc., and the color of the object as well as other minor causes; these combined are easily seen to present a complicated problem which it is impossible to solve completely by rules, or to predict with any definiteness whatever. He who attempts to render a subject in full light and shade can only expect an approximation, as near to the truth as his ability and experience in drawing will permit him to reach, whereas if he seek simply for a suggestive effect of solidity, by the use of shadow, it will be found that there are certain principles underlying the effect, more or less rigid, which will be a help to this end. So that in what follows no pretence is made to completeness in the study of light effects, only a consideration of well-known principles as applied in graphical methods will be attempted.

132. Modifications in Effect due to Character of Surface.—All substances act more or less as mirrors, that is, they reflect to the eye some of the light which falls upon them; the light-rays which are not reflected are absorbed by the surface. If a surface is comparatively smooth, it will reflect the light-rays more nearly parallel, hence more intensely. If the surface be smooth to the extent of being polished, it will reflect even more intensely, and be like a glass mirror, which, for all practical purposes of illustration, reflects all the light-rays, and, as we know from observation, even the image of the source of light may be seen in it. In proportion then as a surface becomes a complete reflector, it will reflect the light more intensely and lose its own distinctive color; a perfect reflector loses entirely local color. An example of this can be seen in the reflections from the brightest spots on polished vases of different colors, even those that are very dark; the light will be practically the same color in each, and an image of the blue sky and of the intervening window-sash may be seen within them.

Again, it is to be noted that a rough surface will give a more uniform effect of color or shade value than a smooth one; this depends primarily upon the point of view, which will be discussed shortly.

If a surface is flat, it will reflect light fairly uniformly throughout, while if it is curved, it will not, the light being reflected from the different parts of the surface differently owing to the angle at which the light strikes the surface and the angle at which it is reflected to the eye, and the relation of the one to the other.

133. Modifications in Effect due to the Direction of the Light.—Light falling upon a surface from a given direction will be reflected in different degrees of intensity according to where the observer is when looking at that surface. If he is standing in such a position that a line of direction from his eye to the surface makes the same angle with it as do the imaginary lines of the light-rays, then the maximum amount of light will be reflected to his eye that is possible with the given character of the surface and strength of light source. In the language of physics, this is the position where the angle of incidence, of the light, is equal to the angle of reflection.

It can easily be seen that if the surface looked at be a curved one, the region in which this condition of affairs is possible is a limited one, and according to the degree of curvature of the surface may be large or small. When small it is what is known as the *brilliant point* or region of highest light. This region is not always possible to ascertain practically, on certain curved surfaces because of the intricate construction which would be necessary; a close approximation to it can be made, however, in some cases, to be instanced later, by taking as the brightest part that in which the rays of light are normal to the surface.

The condition under which the light is reflected least from a surface is theoretically when the rays of light are tangent to the surface.* While this is often observed in practice, it is not strictly true, for there are other modifying conditions, the refraction † of the light for one thing, and the conditions of reflection from surrounding objects another.

The condition of tangency, then, of the light-rays to a surface results in an effect of shadow, what is often erroneously designated as an absence of light.‡ The shadow is a relative matter depending for its value upon the degree of intensity of the light among other things, and may be anything in a drawing from a very light tone to a very dark one.

Again, if a surface be a curved one and the light be tangent to the surface at a certain part, theoretically there will be, if not a point alone, a line or element along which the light-rays are tangent; this line, however, while convenient to consider for purposes of demonstration, never exists, the lighter parts of the surface always blending into the darker by more or less easy stages of shade difference.

Further, the shadow region, either upon a surface or that cast by the surface, will rarely be uniform in tone. It is modified in various ways, the reflection of the light from surrounding objects, as before mentioned, being the chief one. This last will be discussed a little later.

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* For in this position there could be no component of the light-rays normal to it.

† The refraction of the light means the bending of the light-rays out of their path, caused by their cutting across an edge of a surface, or when the light passes from one medium into another, for example from air to water, as can be seen by the distortion of objects partly in and partly out of the water.

‡ It is very difficult to realize such a condition as an absence of light. If a region is not lighted directly, it receives more or less reflected light-rays from other surrounding objects; night is not an example, therefore, of absence of light.

134. Modifications in Effect due to the Point of View.—That the point of view of the observer governs the value of the tone upon a surface follows naturally from what was said in the preceding section.

Given a particular flat surface and lighted from a certain direction, it can be seen, by taking a concrete case, that as the observer moves away from the position before spoken of as the region of greatest reflection of light, i.e., from where the angle of incidence is equal to the angle of reflection, the surface will take on a gradually increasing tone of shade value, so that, no matter how strong the source of the light, as he looks approachingly tangent to the surface the value is noticeably darker than from any other point of view. The change in the observer's position may give a result somewhat analogous to the change in direction of the light-rays when his position is fixed.

135. Modifications in Effect due to Color.—It is not purposed to enter into an analysis of color effects in general, but several points with regard to their modification of the value of a shadow or light are worthy of notice.

Local color may be briefly described as the color of the pigment involved in the coloring of a surface when under ordinary daylight and also when unaffected by any surrounding and modifying conditions.

Apparent color may be defined as that color which a surface appears to be when seen through a medium other than daylight, or the medium of a considerable intervening space of air, or as modified by the presence of other colors, whether merely by contrast or as reflecting light upon the surface.

The apparent color, intimately related to the local color, is that with which we usually have to deal. In general, the darker the local color, the darker the apparent color, and also the darker will be the tone of the surface, if not otherwise modified, of course, by the condition of smoothness. Some colors reflect more light to the eye than others; but while color has considerable influence upon light and shade effects generally, it is best, in sketching in light and shade merely, to ignore its modifying conditions as much as possible. It is the artist's business when using color to represent its effects; not in general when using what is known as a black-and-white medium.

136. Sources of Light. — There may be two general sources of light, *natural* and *artificial.*

The sun furnishes the first. It is so infinitely distant, and withal so brilliant, that what we realize is practically one vast diffused light and its light-rays may be considered as practically parallel. The cast shadow of a surface under the light of the sun, owing to the diffused character of the rays, will be very much modified by reflection and refraction together, principally the former, so that the shadow will not be sharp in contour, as can be verified by observation. This is too often overlooked and shadows represented, sharply and rigidly if at all, and as if the source were very concentrated and near by. The sharpness of shadows cast through the medium of the sun's rays is only pronounced when the surface receiving the shadow is very close to that casting it.

Under artificial light may be included all other sources of light. Artificial light as can be seen, coming from a finite source, has its light-rays divergent, so that the probable form of a certain cast shadow cannot readily be foretold; it will depend largely upon the proximity of the light source; artificial light moreover, not being so brilliant as that of the sun, gives shadow forms that are sharper and not so much modified by reflection or diffused light. In drawing the shadow forms of an object under artificial light, a point is generally taken as the source, from which to draw the divergent rays.

137. Direction of Light-rays.—In descriptive geometry natural light is assumed, and one direction for that of the rays. This seems necessary in order to give both uniformity and simplicity of treatment. Uniformity of treatment is desirable because drawings are read by means of a certain conventional accentuation of lines, and they can only be understood thoroughly when the direction of light is known. And a direction is assumed which is very simple of treatment, namely, that of the diagonal of a cube. If the diagonal is projected upon either face of the cube, it becomes the diagonal of that square face, and since such a figure as a cube or a rectangular solid is generally so placed that its faces are either parallel or perpendicular to the planes on which they are projected, the shadows are rendered extremely simple of treatment.

In perspective, however, there is no uniformity of agreement as to the direction of the light-rays, except to a very limited extent, that is, they are usually taken as coming downward at an angle of 45° to the horizontal plane from some point above and to the observer's left side. The vertical plane, then, which may be conceived to be passed through the rays may be parallel, perpendicular, or at any intermediate angle to the P.P.* Objects are not in general put in one certain relation to the P.P., by draftsmen, and since perspective of light and shade is more or less picture-making, that direction is taken for a particular subject which will produce the best effect. There is no particular light and shade effect which can be said to be universal, but perhaps the nearest approach to it is one in which there is a fairly even distribution of light and shade masses. Strong contrasts with even distribution seem to show best the full effect of form in any object, and this is what is usually sought for in a sketch.

138. Classification of Shadows and Effects Discussed.—Shadows may be divided into two principal classes: (*a*) *Surface shadow*, which is the value of a shadow due entirely to the direction of the light-rays or the position of the observer or both; (*b*) *Cast shadow*, caused by the interception of the light by an object, thus preventing it from falling directly upon other objects.*

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* This general condition will be taken for granted in all demonstrations from now on, and possible directions of light-rays spoken of as coming within these limits unless otherwise mentioned.

The general effects of surface shadow have already been discussed somewhat, and will be again, in connection with different solids. If a surface is flat and it be that part of an object which is turned away from the light or excluded from the light-rays by the substance of the solid, then it will appear fairly uniform in its depth of tone; any modifying conditions, such as the reflection of light from surrounding objects, if the objects be not too near, will be likely to affect more or less the whole surface, lessening the intensity of the shadow. Possibly if the surface is large and the light strong upon some adjoining surface, there may be greater intensity of shadow effect near that edge separating the surfaces one from the other. This may be a real effect or simply an illusion due to contrast, it is real if some other near-by object has an opportunity to throw more reflected light at that part of a surface most removed from the light-mass. This effect has given rise to a general artistic principle of shadows that the deepest shadow lies nearest the brightest light.

If a surface is rounded like a cylinder, a sphere, or a cone, then, whether it be appreciably near other objects or not, there is always present to a greater or less degree the phenomena of reflected light before spoken of. This reflected light is present as a softening of the contour region of the object which lies in shadow. It may be quite pronounced when some brightly reflecting surface is quite near to it, and it may even be possible that strong color effects in this reflected light are noticeable, reflected colors similar to those of the reflecting surface. In the absence of models to draw from, care should always be taken to give some value to this reflected light, as the effect of roundness of an object is very much aided by it.

All objects cast invisible shadows through space. The shadows become visible when they are intercepted by some more or less opaque surfaces.* A striking illustration of this can be had by noticing the faint, almost invisible, shadows of a window-sash with the sun shining in the window when the air in the room is full of dust-particles; these shadows become sharply visible where they strike the floor.

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* The phenomena of surface and cast shadow are so interrelated that it has not seemed advisable to make a division of the subject by separating the consideration of them.

The color of an object casting a shadow does not affect the intensity of the shadow; therefore, of several objects all equally opaque to the light-rays, no one of them will cast a darker shadow than the other. As before mentioned, however, refraction and diffused reflected light do modify the conditions very much. Refracted light softens the sharpness of the edges of a cast shadow, and reflected light modifies very much the intensity of the shadow.

Theoretically an object lying in the shadow of another object cannot itself cast a shadow, while practically it is well known that this is not the case. There may be so much neighboring reflected light that quite noticeable shadows may be cast by objects within shadow.

The shape of the invisible or received cast shadow of an object will vary of course with the shape and position of the object casting the shadow and the shape and position of the object receiving it.

Of several different surfaces receiving a cast shadow one may show a darker shadow than another, due to a difference in character of the surface or a greater depth of local color even in the absence of other modifying conditions.

It is not very difficult in general to foretell the shape and existence of a definite surface shadow, but it is otherwise with cast shadow, and therefore the principal part of the problems in the perspectives of shadows will be the solution of problems as to the shape and position of cast shadows.

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* Shadows, for example, are not visible upon perfectly clear water, but are partially so on muddy water, though the shadow may be further visible on an object under the water.

139. Possible Shadows Cast by a Point, Line, Plane, etc.—(*a*) A *geometrical point* can cast a line of invisible, and a point of visible shadow, since only a single ray of light can be passed through the point.

(*b*) A *geometrical line* can cast a line of invisible and a point of visible shadow, or a plane of invisible and a line of visible shadow. It will cast a point of visible shadow when the line lies in the direction of the rays of light, i.e., is tangent to them. It will cast a line of visible shadow when it lies in any other position with respect to the light-rays.

(*c*) A *limited plane* can cast a plane of invisible and a line of visible shadow, or a solid of invisible and a plane of visible shadow. It will cast a line of visible shadow when the plane is tangent to the rays of light. It will cast a plane of visible shadow in any other position.

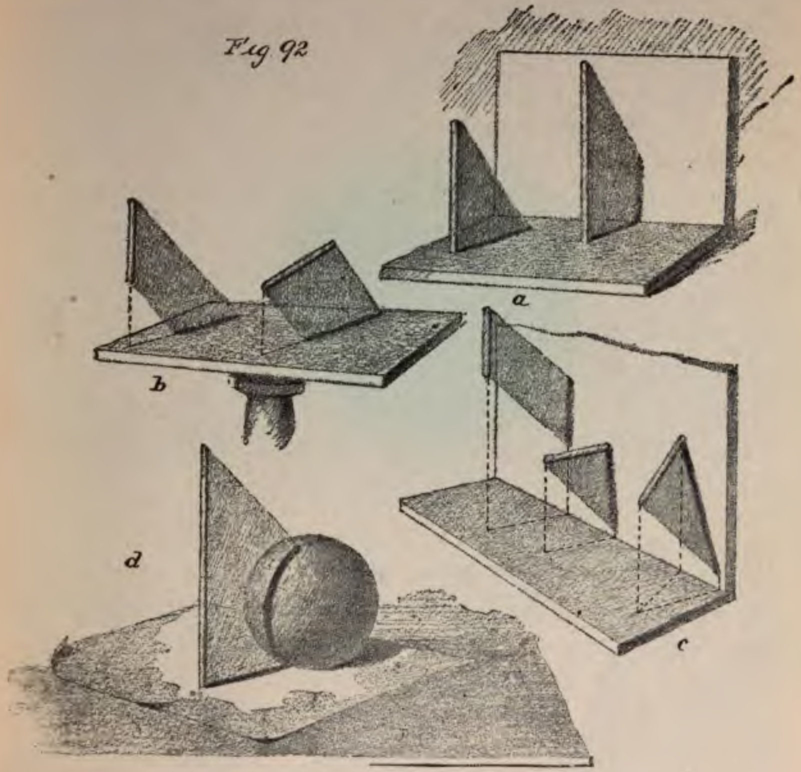
(*d*) A *solid* will always cast a solid of invisible and a plane of visible shadow, the shape of which will depend upon the position and number of edges of the solid casting the shadow contour.

140. To Find the Shadow of a Point or a Line upon a Surface.—To find the shadow of a point upon any surface a ray of light,—as a line,—is conceived to be drawn through the point, and where this line cuts the surface will be the shadow of the point. But the intersection of a line with a surface involves passing a plane through the line and cutting the surface; the line will cut the surface in the line of intersection of the two planes; this was discussed in section 124, and it was there pointed out that it was desirable to pass such a plane as would cut the surface in a line which was easy to draw, a straight line if possible.

Points or lines are generally associated with some object, but it will be well to analyze somewhat further the shadows of a line, as leading up to the consideration of the shadows of solids.

Fig. 92 shows the shadows of various lines upon a horizontal and a vertical plane respectively. Here a vertical plane in each case was conceived to be passed through the line, and a portion of its straight-line intersection with the other plane taken as the shadow. The plane of the invisible shadow is also represented by a light tone. In *a* it can be seen that this invisible shadow takes the form of a triangle, one edge being coincident with the line, one with the surface receiving the shadow, while the hypothenuse is the line of direction taken by the invisible shadow of the uppermost point of the line. In the second illustration of *a* this triangle is seen to be cut off by the interposition of a vertical plane, hence the shadow of the line falls partly on that. It can also be seen that if the vertical plane had been brought nearer to the line the shadow would have extended higher on the plane and there would

Fig 92

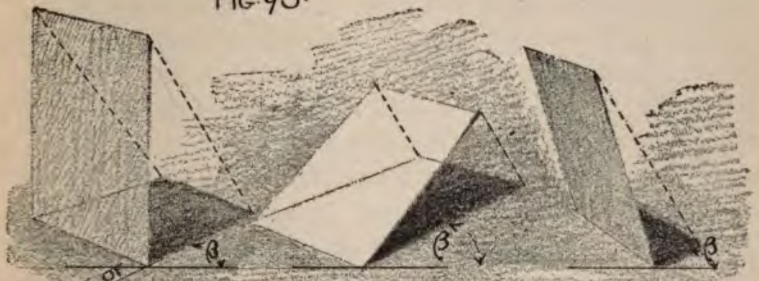


have been less of it visible on the horizontal plane. The problem of the shadow is one of intersecting planes.

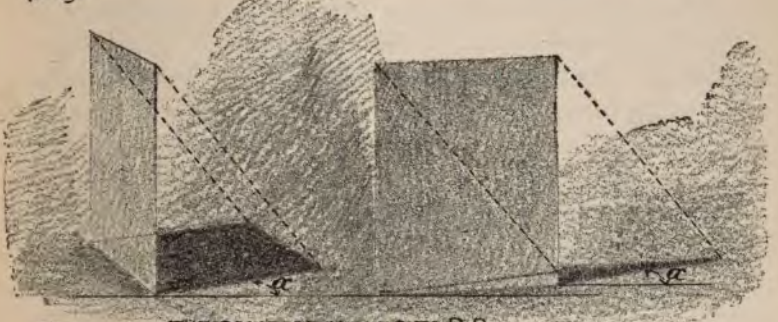
In *b* a vertical line is shown situated entirely above the horizontal plane on which it casts its shadow; the construction shows, however, that it is dealt with exactly as the vertical line in *a* was, and only that portion of the intersection of the plane of invisible shadow with the horizontal plane preserved which was the actual shadow of the line. An inclined line is also shown in *b*, so placed that its plane of invisible shadow is a vertical one also; here the nearest end of the line was taken to be the extremity of an auxiliary vertical line, and then its shadow cast. In *c*, several lines are shown casting their shadows wholly on a vertical plane; the first one is seen to be parallel to the vertical plane, and its shadow is also vertical; this illustrates a very useful principle of shadow effects, namely: *the shadow of a line on a plane to which it is parallel is equal and parallel to the line itself.* The next is a line perpendicular to the vertical plane, and its shadow is constructed similarly to that of the inclined line in *b*; likewise the next line is oblique both to the horizontal and the vertical plane. In *d* we have an illustration of the plane of invisible shadow of a line cutting a spherical surface in a circle which is seen perspectively as an ellipse and is the visible shadow of the line.

141. To Find the Shadow of a Plane upon a Surface.—The method of finding the shadow of a plane is to derive and then connect the shadows of the several edges of the plane. These will form a closed figure or a plane of cast shadow whose shape, it will be found, is determined by the position of the plane with respect to the light-rays and the surface receiving its shadow. Fig. 93 illustrates this by simple rectangular planes. An examination of the construction will suffice to show how the shadows were obtained. The directions of the shadows on the horizontal plane both in these and the preceding figures were assumed arbitrarily as well as the downward direction of the light. Parallel planes having parallel traces on a common plane, and the light-rays also being parallel, are,

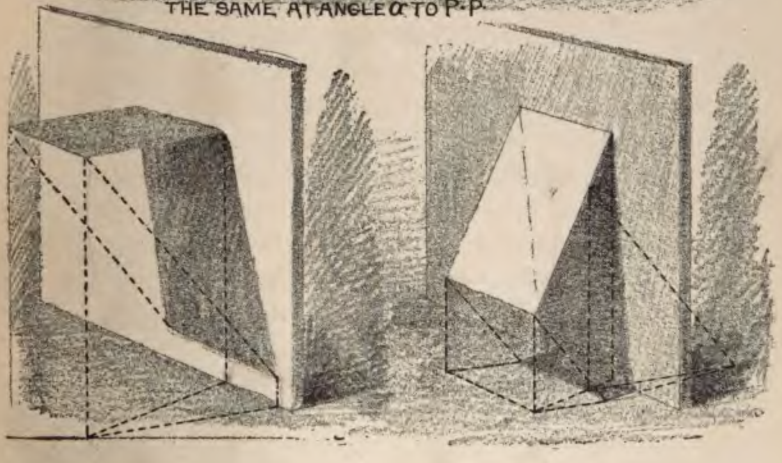
FIG. 93.



INVISIBLE PLANE OF SHADOW AT ANGLE β TO P.P.



THE SAME, AT ANGLE α TO P.P.



by an observation of the perspective convergence of such lines, sufficient data to enable one to work out the shadows of any of the simple forms upon a plane surface. If a plane casting its shadow is present, having a curved contour, then it is only necessary to consider a sufficient number of points in its perimeter which, by their shadows, will make it possible to draw a fairly smooth curve of shadow through them.

It needs to be further noted, in working out the shadows of these simple plane forms, that, as observed in the preceding section, when there are lines or contours entirely separated from the surface receiving the shadow, their several points should be taken as being the extremities of imaginary lines extending perpendicular to the surface receiving the shadow. In this way the triangle of the invisible plane of shadow in each case, being perpendicular to the receiving surface, can be easily sketched, that part of it only being preserved which is cast by a real line or edge.

142. To Find the Shadow of a Solid upon a Plane Surface.—The method of finding the shadow of a solid upon a plane or any other surface will appear clearer later on, but a few introductory words may be said about it now, while discussing the shadows of lines and planes. The shadow of a solid is effected by casting the shadows of its several edges, just as with a plane; however, all the edges will not cast shadows; some will lie wholly in the light or wholly in the shadow. Only those edges which lie between the lighted and the shaded portions of the solid need to be considered.

In the absence of better data, the shadow of each edge may be worked out and those shadow-lines which enclose the largest area of surface will be the ones bounding the uniform flat shadow of the solid.

Other data, however, *are* in general resorted to. By drawing a series of several planes, which may contain rays of light, tangent to the solid at different parts, it can be found that those edges or parts of the solid at which these planes are tangent will be the edges, real or imaginary, which cast the contour of the shadow. This will be discussed more fully later. Inspection also will very often enable one to tell which will be the edges casting the shadow contour. The edges of a subject which cast shadows, taken together, constitute what is generally known as the *line of shade* of the solid.

143. Linear Perspective Construction of Shadow Effects.* —Assume, as was pointed out in section 137, that the light-rays fall at an angle of 45° to the horizontal plane from some point above and to the left of the observer. The locus of the vanishing-points of the light-rays, as they are assumed to come from various sources, would be the curve of the lower nappe of the hyperbola as described in sections 108 and 116, and it probably can be readily seen that when the light-rays are parallel to the P.P. and hence to the asymptote of the hyperbola, the simplest construction results, both in linear perspective and in sketching.

Fig. 94 shows the simple case of the shadow of a solid upon the horizontal plane when this direction is assumed. The vertical planes passed through the vertical edges of the block have intersections with the horizontal plane in lines parallel to the P.P. It is true that shadows do not always fall upon a horizontal plane, but it will be found by experience that this direction of light-rays *does*, in general, give simplicity of construction; in addition shadows on a horizontal plane somewhere in a picture *do* form a principal part of cast shadows of most subjects, for example those having a horizontal base to stand upon.

Fig. 95 shows the state of affairs when the light-rays are assumed to come from such a direction that a vertical plane which may be passed through and containing a ray stands at

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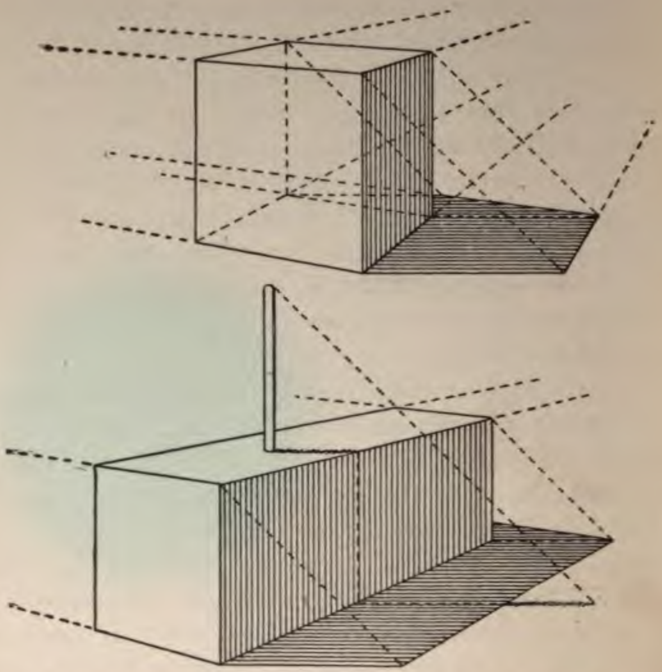
* It is to be supposed that by this time the student has such a mastery of perspective principles that he sees readily their connection with sketching. Therefore it hardly seems necessary after this to make the linear perspective processes take precedence of the approximate methods. In sketching shadow effects, one should be largely free from restraint, and as a matter of fact in general practice shadows are rarely worked out by exact methods.

an angle of 45° to the P.P. $V.P.^3$ is found as one point i
the vertical trace of this plane. Rotating the observer abou
this trace to $S.P.^2$ makes it possible to get $V.P._L$ as th

FIG. 94.



vanishing-point of the rays of light at 45° to the horizontal plane. Vertical planes of light passed tangent to the block will have horizontal traces vanishing in $V.P.^3$ The portion of these traces which becomes visible shadow is found by drawing rays toward $V.P._L$ to cut their several traces.

If the cast shadows did not fall upon the horizontal plane but upon some other inclined plane, then the line of intersection would have to be found between the plane receiving the shadow and the plane of light passed tangent to the object, in order to get the vanishing-point of the parallel traces, which latter in this case would not be horizontal.

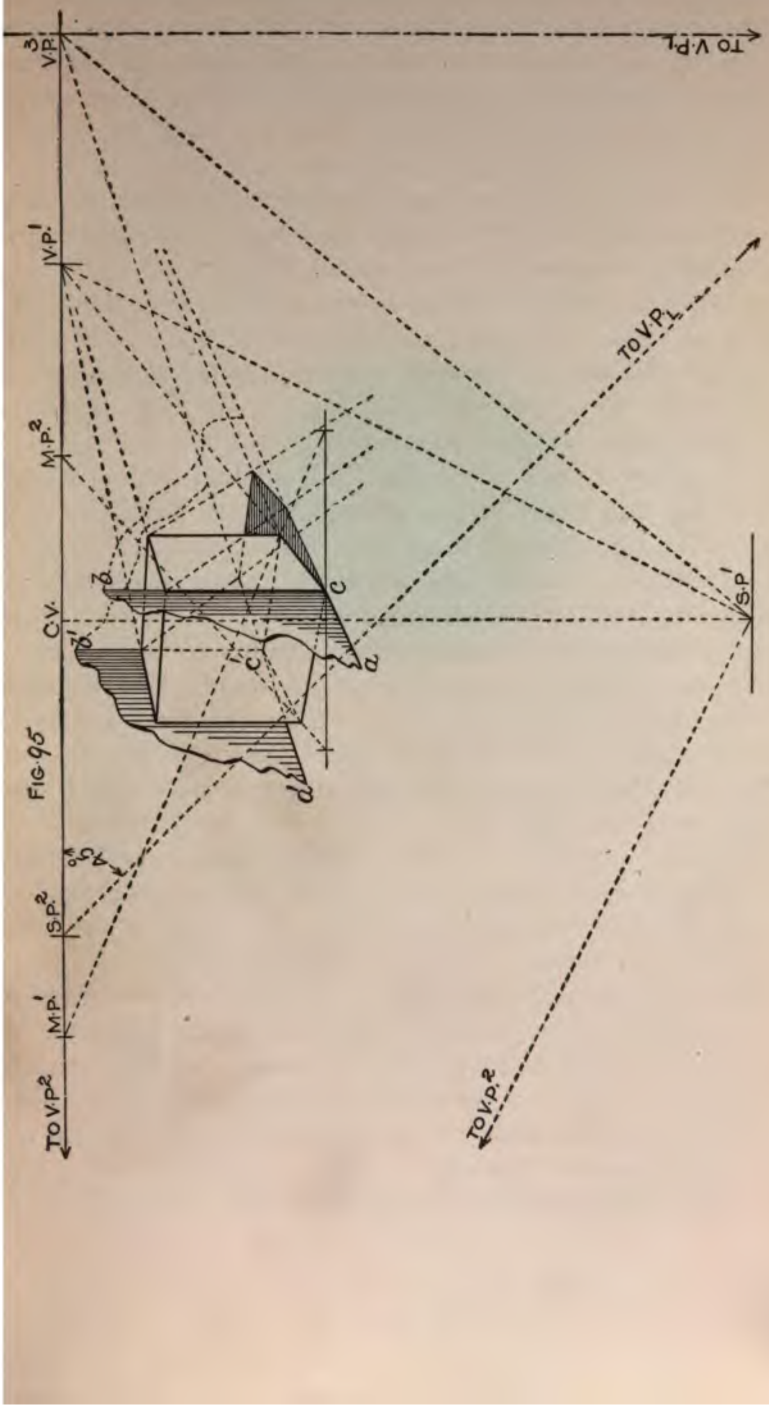


FIG. 95

A general relation which exists between a line and its shadow on any surface and which will help in the solution of problems may be stated as follows: *The visible shadow cast by a line upon a surface has its vanishing-point at the intersection of the vanishing-traces of the plane receiving the shadow and the invisible plane of shadow of the line.*

It is well to note before proceeding that the planes *abc* and *a'b'c'* passed tangent to the block gave immediate knowledge as to those elements which cast visible shadows.

Hereafter sketch solutions will prevail where possible, and only the linear perspective methods touched upon if there is sufficient novelty of construction to render the investigation profitable.

144. Miscellaneous Problems.

Problem 1. Sketch an equilateral triangle standing vertically upon one of its edges, its plane at about 70° to the P.P. Cast its shadow, rays parallel to the P.P.

Problem 2. Sketch a cube standing on a face, edges equally inclined to the P.P. Sketch its shadow, rays parallel to the P.P.

Problem 3. Sketch a square frame similar to that in Fig. 69. Place it on an edge inclined at 30° to the P.P., the plane of the square at 60° to the H.P. Sketch the cast shadow on the horizontal plane, and the surface shadow on the object; light assumed to fall in the vertical plane which makes an angle of 45° with the P.P.

Solution.—Since the edges casting shadows are inclined to the horizontal plane, work out the shadow of points or corners regarded as the extremities of vertical lines touching the horizontal plane. The remainder of the process does not differ from the preceding problems except that there will be a convergence of the horizontal traces and of the light-rays.

145. Possible Shadows due to Changes in Direction of Light.—It was seen in earlier chapters how a profitable investigation could be made by studying the changes in form of simple objects due to changes in position; similar investigation of shadow phenomena will be found to be also profitable. The changes being in the assumed direction of the light-rays, and the position and form of the object receiving the cast shadow.

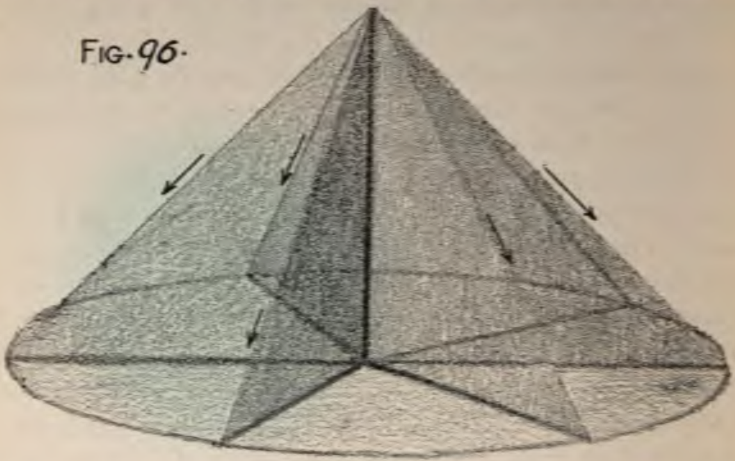
Beginning with a point, then, it is evident that the only change worthy of notice in its shadow would be that due to the distance of the point from the object receiving its shadow. No matter from whence the direction of the light, the shadow of the point would approach coincidence with the point itself as the plane receiving the shadow approached coincidence. Hence the distance of a receiving surface may, in a measure, be made apparent by the distance of the shadow from the point casting it.

146. Possible Shadows of a Vertical Line on a Horizontal Surface.—For the sake of simplicity assume a vertical line touching the horizontal plane and of certain fixed length. Two variables are present, the direction of the light-rays with respect to the horizontal plane and with respect to the picture-plane. The shadow will in all cases extend from the foot of the vertical line to some point governed by the height of the light source above the horizontal plane. Its length then will vary from infinity as the light-rays are coincident with the horizontal plane, i.e., as the sun is appearing in early morning to zero when the light-rays are directly above the line, theoretically at noon. When the rays fall at 45° to the horizontal plane, then, the cast shadow will be the picture of a line as long as the line casting the shadow, and, as can also be seen, the plane of the invisible shadow is the picture of a right isosceles triangle with a 45° base angle. If the light fall at other angles, the length of the shadow can be found from simple geometrical construction. It is not as profitable to discuss the effects through changes in direction of the light-rays to the horizontal plane as it is through the changes with respect to the picture-plane, together with the changes in position of the object or surface receiving the shadow.

Consider the case of the vertical line as shown in Fig. 96, light-rays at 45° to the horizontal plane but variously inclined with respect to the picture-plane. The invisible plane of shadow of the line is in all cases, of course, a vertical plane, so that its angle with the picture-plane is easily estimated. It is convenient for this reason in most problems in shadows to develop the shadow by using vertical planes passed through the rays of light or, next in simplicity, planes perpendicular to the receiving surface.

When the light-rays are parallel to the picture-plane the triangle of invisible shadow is not foreshortened. As they are assumed more and more inclined to it, the line of cast shadow becomes increasingly foreshortened, till the plane of invisible shadow is perpendicular to the picture-plane. A slight reflection upon these changes, as shown in the figure, leads to the conclusion that the shadow of the top of the line travels in a circle on the horizontal plane, its picture being an ellipse and analogous in its conditions to the directed plane described in Chapter VI.

FIG. 96.



The value of this investigation is made apparent by a moment's reflection. If it is desired to cast the shadow of any line when the exact direction and length of the shadow are not at once evident upon inspection, then some position and direction may be used tentatively which *is* apparent and simple of construction, and from it, by an observance of such a law of change as was shown above, the true shadow may be readily derived by comparison.

147. Possible Length and Direction of the Shadow of a Line on any Plane Surface.—If a line is taken of definite length and fixed position, and the light-rays also assumed as fixed, the changes possible in its cast shadow will be changes due to the form and position of the plane receiving the shadow. For simiplicity assume a vertical line as before and a flat surface receiving its shadow, first coincident with the horizontal plane, then rotated through various angles with respect to it and the line. These conditions are expressed in Fig. 97, the light-rays being assumed parallel to the picture-plane. The receiving plane is rectangular in shape, α being the angle which one horizontal edge makes with the picture-plane; the adjacent edge *bc* is taken as the axis of rotation. *oa* is then the shadow of the vertical line on the horizontal plane found as in the preceding example; *oo'a* is the triangular vertical plane of invisible shadow. Suppose now the receiving plane moved to the position 2: the visible shadow on it in its new position will be a portion of the line of intersection with it of the invisible plane of shadow. To get this intersecting line use is made of the fact that the line of intersection of two vertical planes is vertical, hence the plane in position 2 will have p' vertically over p, as a point in the line of intersection of the two planes, for *bp* moves in a vertical plane about *bc*. The cast shadow in position 2 will begin at *d* and extend to where the line *o'a* cuts the line of intersection of the planes, namely, a'. In like manner any other position of the receiving plane can be located, such as position 3, by means of p''. When the receiving plane takes a vertical position the cast shadow would be vertical also.

In these various shadows, it is to be noted that as the plane moves nearer to the line the shadow also moves nearer, but an important relation is brought out in this case in addition, giving a solution to the question in what way does it grow nearer. As the receiving plane moves through 90°, the

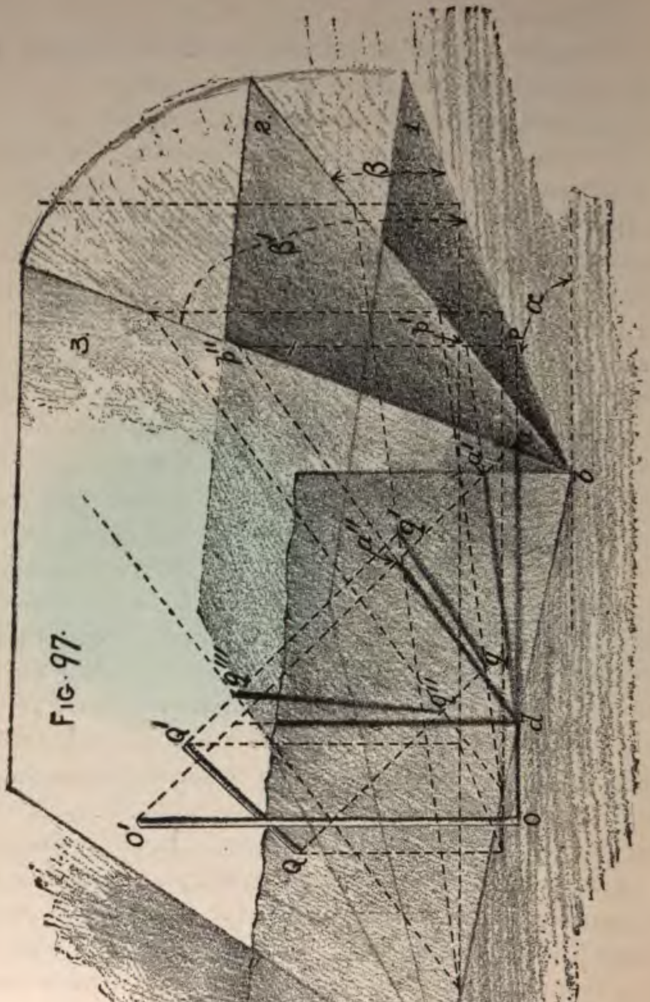


FIG. 97.

shadow line moves through 90°, though not proportional to the angles through which the receiving plane moves. They are dependent upon the angles which the line of intersection of the invisible shadow plane and the receiving plane make successively with the horizontal plane. If the angle in any specific case of this line of intersection with the horizontal plane can be judged, then the solution is simplified; because, if the plane of the invisible shadow were oblique to the picture-plane, then the directed plane could be used by which to estimate the value of the angle, a line similar to *bc* being the axis of the directed plane.

It can also be readily seen from this problem that had the receiving plane been slightly concave toward the line the shadow would have been curved and concave toward the line and *vice versa.*

From these observations may be deduced several very useful principles, useful in anticipating the forms of cast shadows: (1) *As a surface or part of a surface receiving a shadow nears the object casting it, the shadow nears the object also.* From the observed direction of the shadow when the receiving plane stands vertical, we have another illustration of the principle mentioned in section 140, namely, that the shadow of a line on a plane to which it is parallel is equal and parallel to the line itself; also from this the principle can be extended likewise to include any plane surface. (2) *The shadow of a plane upon a plane to which it is parallel is equal and parallel to the plane itself.* Perspectively, of course, the edges of the plane and the parallel lines of its shadow converge in general to vanishing-points. That the above principle must be true follows from the fact stated in the preceding chapter, that when two parallel planes are cut by a third plane, the lines of intersection are parallel.

The cast shadow is one plane, and the original is another; the invisible planes of shadow of the several edges cut these two in parallels respectively.

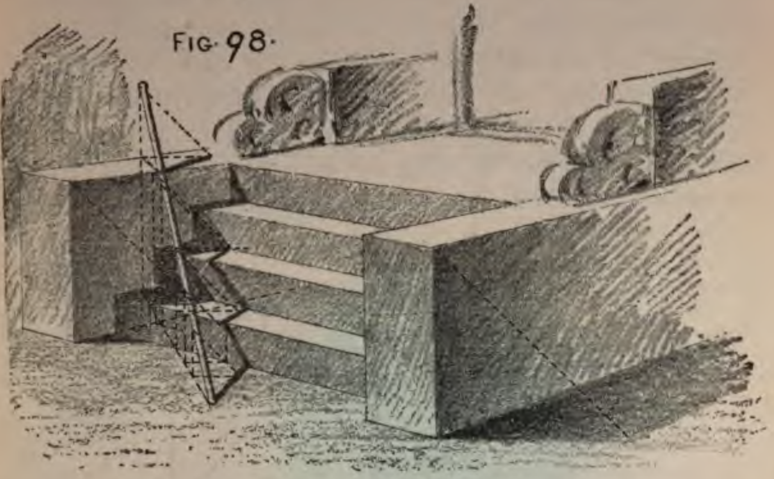
If the original line in the former example had been taken oblique to the horizontal and picture planes, then its shadow could be determined for any position of the receiving plane by using two invisible planes of shadow passed through the extremities of the line, and finding by means of these the shadows of the points on the receiving plane and connecting them. This is shown in the figure for the shadows of the line QQ' upon the planes 2 and 3. qq' is the shadow upon the plane 2, $q''q'''$ is the shadow upon the plane 3. The horizontal projection of the line QQ', shown dotted, was necessary in order to pass the vertical planes through its extremities.

148. The Least Shadow of a Line, Plane, or Solid.—If the light-rays are fixed, and the given line is fixed in any other position than tangent to the light-rays, its least shadow upon a plane surface will be when the surface is perpendicular to the light-rays. This follows logically from the geometry of the conditions. The invisible shadow of a line, for example, is a plane bounded by two parallel edges. The shortest distance between these two parallel edges is a common perpendicular to them; similarly, this demonstration may be extended to include the shadow cast by a plane or a solid. In connection with the last, it also deserves to be noted that the least visible shadow will be when the least number of edges of the solid are casting visible shadows.

149. The Shadow of a Line upon a Broken Surface.—Fig. 98 illustrates the case of a line casting its shadow upon several surfaces, a series of horizontal and vertical planes respectively. No general law of effects can be derived for such a case further than was done in section 140. These, however, are exemplified by construction in the illustration.

When the shadow of a line traverses several plane surfaces it is only necessary to establish two points in the line of the shadow on each surface. If the shadow extend entirely across the surface, any two points will do; if not, then one of the points should be the shadow of the extremity of the line. If the shadow form on the various planes can be judged by inspection, then only the shadow of the beginning and ending of the line need be found.

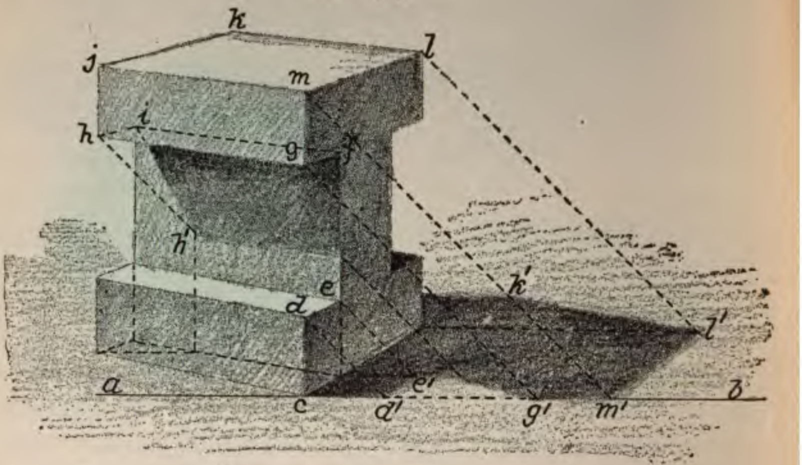
FIG. 98.



150. Miscellaneous Problems in Surface and Cast Shadows.—Following are a few problems worked out in part and illustrating the principles of surface and cast shadow which have been discussed:

Problem 1. Fig. 99 shows a T-shaped block, namely, a block consisting of a central plate terminating in horizontal upper and lower flanges, light-rays are assumed parallel to the picture-plane. The shadow of a vertical line would take the direction, on a horizontal plane, of the line *ab*, and all the points used to cast the shadows of the various edges may be thus considered as the extremities of vertical lines. The first thing to be done is to ascertain which edges will cast visible shadows. A tangent plane to the solid along the edge *cdgm* and containing a ray of light, i.e., a plane parallel to the picture-plane, will at once show that the face *cml* is in shadow and *cmj* in light; similarly the face *hjk*, parallel to *cml*, would be in light, and the rear face, parallel to *cmj*, would be in shadow. Those lines or portions of lines separating these lighted from shaded faces will cast visible shadows; the following are these edges: cd, de, part of ef, gh, hi, gm, ml, lk, and the rear vertical edge through k, concluding with a very small portion of the shadow of the lower horizontal edge of the upper flange. The corresponding shadows of various points are similarly lettered and are obtained as follows: cd casts the horizontal line of shadow cd', and all edges similarly directed and casting their shadows on the horizontal plane are constructed in the same manner by using the hypothenuse of a 45° triangle as before described. de casts the shadow $d'e'$, and is easily

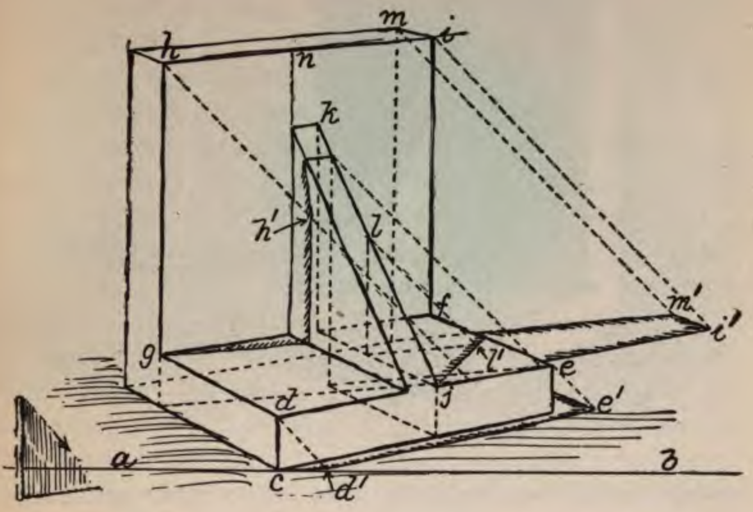
FIG. 99.



located by the shadows of its extremities, or, having that of one, the shadow may be sketched to converge to the proper vanishing-point. The shadow h' of the point h is found by passing a vertical plane through h and parallel to the picture-plane; h' is a point in its intersection with the plane efi. Part of the edge hg casts its shadow on the plane fei, and part on the horizontal plane; in either case the shadow is parallel to the edge casting it and is easily located. The shadows of ml and lk will be perspectively parallel to the respective edges and also perspectively equal, and might be at once sketched without the use of the invisible line of shadow of the points l and k.

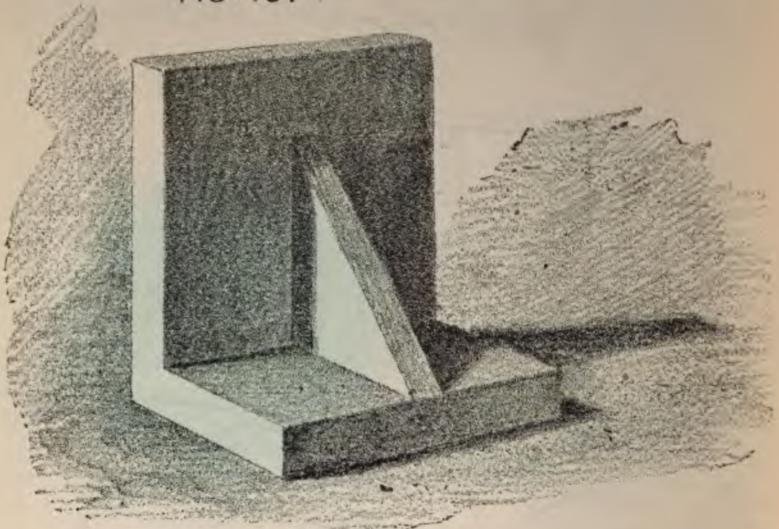
Problem 2. Fig. 100 is an angle-plate having an inclined buttress in the angle. In this subject assume the light to fall so that a vertical plane containing a ray of light would have a horizontal trace in the direction shown by the shadow of the vertical line to the left, which acts as a guide to the shadow effects, and is a convenient device that may be used frequently. This vertical plane is inclined to the picture-plane at a less angle than the faces *cde* and *ghi*. By a similar analysis to that in Fig. 99 it is found that the various edges casting shadows are *cd*, *dje*, part of *fe*, *gh*, *hi*, *im*, the rear vertical edge through *m*, and finally part of the edge *jk*. In the main the shadows are obtained as in the preceding problem, but a few special constructions deserve to be noticed. Part of the edge *hi* casts its shadow on the vertical face of the buttress, part on the inclined face, and the rest on the horizontal planes; that part which falls on the vertical face of the buttress was obtained by conceiving this plane extended to go through the point *n*; this last will then be one point in the

FIG. 100.



shadow, h' the other. To get the shadow of the inclined edge jk, one point l is taken at random and its shadow found. By inspection it can be seen that the remainder of the shadow of jk falls on the horizontal plane behind the subject; if, however, it were visible, the shadow of one point would be sufficient and the line of shadow would be a parallel perspectively through this point to the line jl'. Fig. 101 is added to show how the resulting sketch may be treated after the necessary constructive work has been performed.

FIG. 101.



Problem 3. Take any inclined line and, assuming its inclination by considering its extremities as the ends of vertical lines limited by the horizontal plane, sketch its shadow on the horizontal plane and then on an inclined plane taking successively the positions of 30°, 60°, and 90° with respect to the horizontal plane, rotated similarly to that of Fig. 97. Light-rays parallel to the picture-plane.

Problem 4. Sketch a cube as if standing on a horizontal plane on one of its edges, the edge at 60° to the picture-plane, and the adjacent faces equally inclined to a horizontal plane;

next sketch an inclined plane surface, by using its initial and horizontal traces in such a position as to receive the least cast shadow of the cube, and sketch the shadow. What position should the plane have to receive the greatest cast shadow?

Problem 5. Sketch a hexagonal prism, proportions optional, standing on the horizontal plane on one of its rectangular faces, the edges of the face at equal angles to the picture-plane, that in the hexagonal face vanishing toward the right; next sketch a vertical stick twice as high as the diameter of the hexagonal face, surmounted by a small square plane; place the stick near the prism so that its shadow will fall on it. Assume light-rays oblique to the picture-plane, but not at such an angle as would light the near hexagonal face, and then sketch all the cast shadows and surface shadows.

151. General Treatment of the Shadows of Curves and Curved Surfaces.—It has already been pointed out that the shadows of curves are worked out similarly to the shadows of points and lines, the difference being chiefly that, since a curve has changing direction, its shadow can only be found, in general, by casting separately enough shadows of points in the curve to enable a satisfactory line of shadow to be drawn through them. The shadow of a curve on a plane surface and the shadow of a straight line on a curved surface are treated somewhat similarly, though the latter is generally a more difficult problem, because the intersections of various planes passed across the line and the curved surface require to be found. The forms of the surface and cast shadows of curved surfaces will be treated under separate headings, but before doing so a few general facts regarding shadow phenomena deserve to be discussed.

The shadow phenomena to be considered on any curved surface are the region of brilliant light and of deepest shadow. These have been briefly alluded to in section 133. In the first of these definite position is not in general necessary, hence the compromise suggested of assuming the brilliant region as that in which a line or plane containing the rays of light is normal to the surface; also the exact line of shade or element of deepest shadow is not necessary except sometimes to locate the position and shape of the cast shadow, which is of course that of the line of shade.

A general method of finding the line of shade on a curved surface is to pass a tangent plane containing the light-rays, or a series of tangent planes, to envelop the surface, thus forming a cylinder of rays; the element of contact of the tangent plane or series of planes gives the line of shade. In sketching the shadow of a curved surface it is not usual to make a sharp distinction between the lighted and shaded portions of the surface, if intermediate gradations are dealt with at all; the division is made less pronounced as the effect of gradation is represented more in detail. Faithfulness to or emphasis upon an exact line of shade does not necessarily produce the best effect in shading.

The construction of the line of shade on irregular curved surfaces, such, for example, as that shown in Fig. 68, is a difficult process, and involves so much work that it is better in dealing with the perspective of such forms to approximate the result roughly, basing it upon what the line of shade would be as derived and modified from orthographic projection.

The circular curve, occurring as it does most frequently in practice, will be the only plane curve discussed, and of the curved surfaces only the cylinder, cone, and sphere in detail.

152. The Circular Curve and its Cast Shadow.—The line of shade of a circular plane will of course be the perimeter of the circle itself, and the form of its cast shadow upon a plane surface can be, like the perspective of the circle, one of three forms, a straight line, a circle, and an ellipse. Its cast shadow will be a straight line when the plane of the circle is tangent to the light-rays. It will be a circle under two conditions: first, when the plane of the circle and the receiving plane are perpendicular to the light-rays; second, when the receiving plane makes the same angle with the light-rays that the plane of the circle does but oppositely directed, and the line of intersection of the two planes produced is perpendicular to the light-rays.* The cast shadow will be an ellipse under any other than the above-named conditions. Of course it must be understood that when the cast shadow is a circle it can only appear such when the plane of the shadow is parallel to the picture-plane, and similarly the true form of the ellipse of shadow will only appear when similarly placed, otherwise the picture of the ellipse will be another ellipse of a different ratio of axes.

A graphical way to look at the shadow of a circular plane is to consider its invisible cylinder of shadow, of which the circle is one base and its shadow the other; the axis of this cylinder is the invisible shadow of the center of the circle. If the plane receiving the shadow is parallel to the circular plane, then it is only necessary to get the shadow of the center and sketch in about it an equal figure, making due allowance, perspectively, for the difference in position of the original and its shadow with respect to the station-point. If the receiving plane is oblique to the circular plane, then it is to be observed that the shadows of any two diameters of the circle at right angles to each other will be a pair of conjugate diameters † of the ellipse of shadow, and it would be possible in this way to construct the shadow, though it would not be as desirable a process for use in sketching.

Looking at the cylinder of invisible shadow of a circle for a key to the shadow form helps to a clear conception of the second of the two conditions mentioned in the preceding paragraph, in which it is stated that the shadow of the circle will be an equal circle. The circle, its invisible shadow, and the receiving surface form together an oblique cylinder with bases at equal angles to the axis, but whose planes converge to meet in a line which is perpendicular to the axis. This is a rare

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[*] This is what is known in mathematics as a subcontrary section.

[†] A pair of conjugate diameters of an ellipse are so related to each other that tangents to the ellipse erected at the extremities of one of the diameters will be parallel to the other.

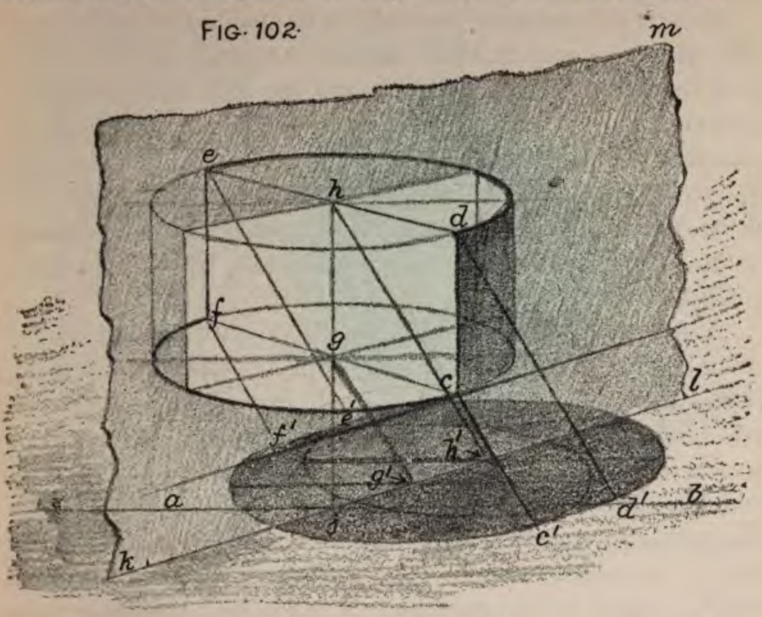
case, however, and is only mentioned in order to state the whole truth concerning the shadow form.

A practical method of sketching most readily the shadow of a circle is to use the shadow of a circumscribing square of the circle; the ellipse of shadow may then be sketched as a curve tangent to the picture of the middle of the sides of the shadow of the square. In some positions and directions of rays of light it may be possible to anticipate the direction of one of the axes, when of course the sketching is much simplified; such a case, for example, would be when the circular plane was parallel to and casting its shadow upon a horizontal plane. In case the circle is parallel to the receiving plane in any position of the latter, the shadow will of course always have its major axis parallel to the initial and vanishing traces of the planes; this fact, where the traces are known, helps to locate the shadow. No illustrations of the shadow of the circle will be here given because the treatment will be taken up in connection with the circular cylinder.

153. The Surface and Cast Shadow of the Right Circular Cylinder.—If a right circular cylinder be suspended in space free of contact with any other body or surface, then its line of shade, i.e., its contour line which casts its cylinder of invisible shadow, will in general, if the light-rays are not parallel to the axis of the cylinder, consist of the opposite halves of the perimeter of each base, and two opposite straight-line elements connecting these curves. It can be seen from this that the visible shadow on a plane surface will consist, in general, of two elliptical curves joined by opposite tangents. If the base of the cylinder is in contact with a plane, part of the curve will of course be its own shadow.

The facts regarding the form of the surface and cast shadow are best studied by taking a very simple case. Fig. 102 is a cylinder with axis vertical and the plane of its base elevated slightly above a horizontal plane. According to the general method for finding the line of shade on any curved surface given in section 142, it is seen that the plane tangent to the surface of the cylinder and containing the light-rays will be vertical. If the direction of light-rays, with respect to the horizontal plane, be for the moment ignored, a normal plane to the cylinder shown at kml, making the angle bjl with the picture-plane, can be assumed as a plane parallel to this tangent plane. Sketching a tangent to the base of the cylinder through c and parallel to this plane gives cd, the element of deepest shadow; opposite to it, fe, will be a shadow eleme hidden by the surface of the cylinder. Connecting these two are the curves, shown by heavy lines, completing the contour *line of shade*. Since the receiving surface is horizontal, only the shadows of the centers of the bases, g' and h', need be found, $c'd'$ and $f'e'$ being afterward located by sketching tangents to the ellipses of the shadows. The light-rays being assumed at an angle of 45° to the horizontal plane, $c'd'$ will be the picture of a line equal in length to cd, similarly with $e'f'$. Also, the major axes of the ellipses of the shadow are the pictures of lines equal to the major axis of the ellipse of a

FIG. 102.



base of the cylinder. Various points in the curve of the cast shadow could have been found if desired by the same method used to get g' and h'. As before mentioned, the element *cd* is only a theoretical element and would disappear entirely in a carefully shaded drawing of the cylinder. Either the direction of the light-rays may be assumed, as in this case, or the element of deepest shadow; this last is a very convenient plan sometimes and will save considerable constructive work.

If the light-rays were assumed to be parallel to the picture-plane it can be seen that the element of deepest shadow would lie in the middle of the cylinder and the high light just at the left-hand edge if the cylinder were directly in front of the observer; if perpendicular to the picture-plane, the elements of deepest shadow would be the right and left contour elements of the cylinder, and the high light would lie along the middle element. In this case no shadow effect, so called, would be noticeable.

A few words may here be in order with regard to the treatment of the graded shading on a cylindrical surface. Theroetically no two consecutive elements of the cylinder will reflect light to the eye with equal intensity, though it will be equally intense along any one element practically, on a short cylinder. However, it is not possible to realize these conditions either to see them on a particular subject or to interpret them in a drawing; moreover, if attempted, there is danger of making such a careful blending of tones that the strength of effect is sacrificed. Primarily the cylindrical surface should have a light-mass and a shaded mass and as much blending between the two as will serve the purpose of the sketch and preserve the effect that is desired.

A somewhat more general case of the cylinder is that shown in Fig. 103, which is one lying on a horizontal plane, its axis oblique to the picture-plane. In order to find the line of shade on it, three alternatives present themselves: (1) To derive it directly by the method of tangent planes; (2) To derive it inversely by casting the shadow of the cylinder first;

(3) To assume at once the element of deepest shadow and thus fix the direction of the light. These will be discussed in the order given.

(1) Fig. 103 shows the constructive work necessary to derive the line of shade directly, and may be thus described: In section 105 it was shown that if two intersecting lines be given, the traces of their plane can be found by connecting the corresponding traces of the lines. In order to pass a plane tangent to the cylinder which contains a ray of light, another line must be located in the same plane and parallel to the axis of the cylinder, a condition which the required plane must also fulfil. Therefore a plane containing the light-rays is first passed through the axis as follows: A line is drawn through any point *c* of the axis in the direction *cd* of the light-rays, to cut the horizontal plane in *d*; this line, prolonged, will intersect the picture-plane in *g*, a point in the initial trace of a vertical plane of light, *L*. The cylinder axis prolonged meets the picture-plane in the point *f*, hence a line through *fg* is the initial trace of a plane parallel to the desired plane. *e* and *d*

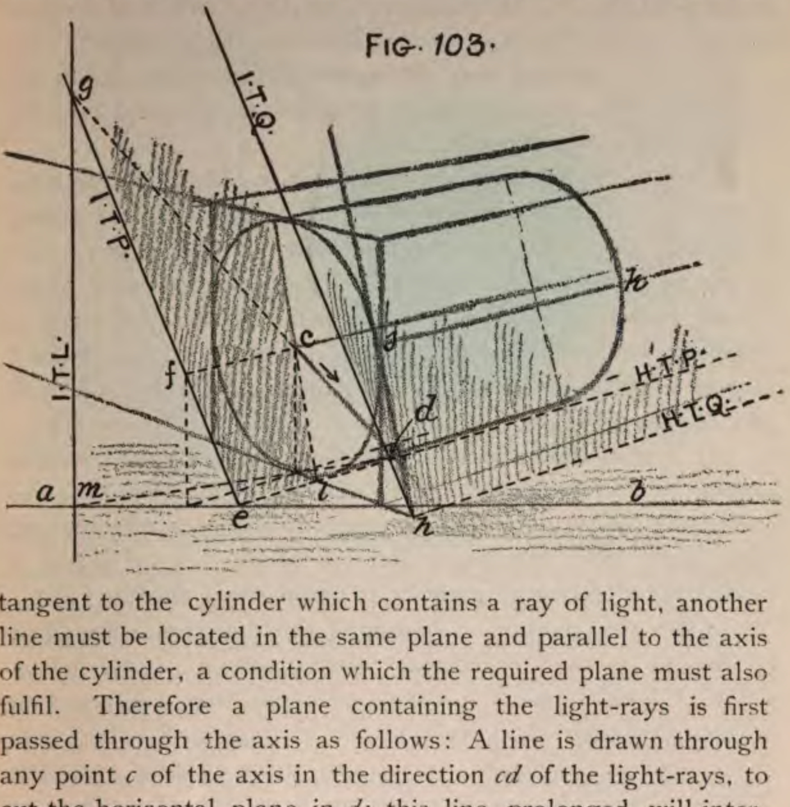


FIG. 103.

must be points in the line of its horizontal trace. A plane parallel to this one and tangent to the cylinder will have correspondingly parallel traces. To get a point in one of its traces, *c* and *l* are connected; the latter is the point in which *de* cuts the plane of the base of the cylinder; the line *cl* is the line of intersection of the plane *P* with the cylinder base; a tangent to the base perspectively parallel to this line will give *h* as its horizontal trace and as a point in the horizontal trace of the required plane *Q*. The element *jk* is the element of deepest shadow, and the traces of *Q* are parallel to those of *P*.

That this method may be made perfectly general in its application for cylinders in all positions, the process may be stated in the following rule:

RULE.—*To find the line of shade on a cylinder whose axis is oblique to both the picture-plane and the horizontal plane, first find where its axis pierces either or both planes, then through any point in the axis pass a ray of light to cut either or both planes. The corresponding traces of these lines will be points in the corresponding traces of a plane of light. Planes tangent to the cylinder and parallel to this plane of light will give the line of shade.*

(2) The method of deriving the line of shade by the inverse process of first finding the cast shadow of the cylinder is of more use in sketching, since it involves less construction, and particularly since the exact location of the line of shade is never desired except it might be to get the cast shadow. Fig. 104 shows this method. Assuming the direction of the light by making a small preliminary sketch shown at the left, a series of several points in the curve of the base are taken at random and their shadows sketched; a tangent to the curve of the shadow and parallel to the axis of the cylinder gives the shadow of the element representing a portion of the line of shade. Hence by working backwards from a point of tangency of this straight-line shadow, that point is found on the cylinder to correspond with it, and from it the element of deepest shadow can be found.

(3) The last method of assuming at once the element of deepest shadow is more particularly applicable when the cast shadow is not to be dealt with except on a surface coincident with or parallel to the base of the cylinder; it puts difficulties in the way of readily anticipating the particular direction of light-rays which would produce the conditions, though they could be worked out by linear perspective. On the other hand, given the direction of the light-rays, it is not easy to tell without construction approximately where the line of shade would come except in very simple cases; such a series will now be illustrated; it is from an analysis of conditions similar to that used in discussing the possible shadow of a line on any plane, where it was shown that by knowing what the shadow would be in some simple instance, approximation could be made to the form under some other condition easily derived from the first.

FIG. 104

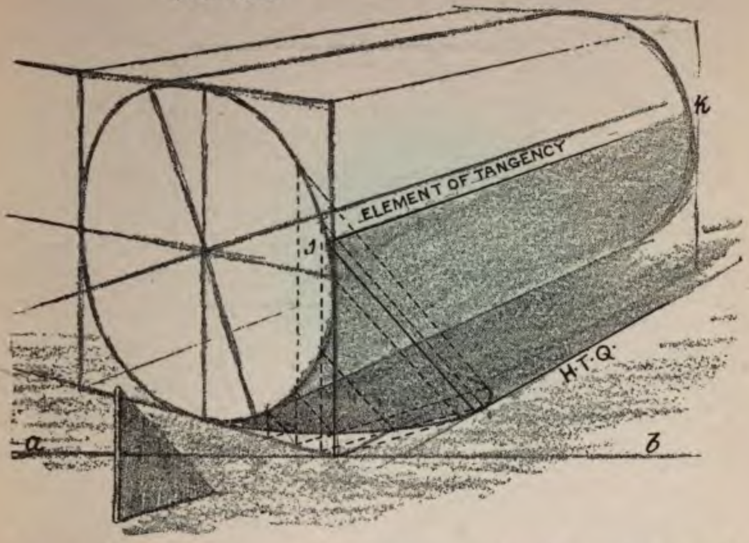
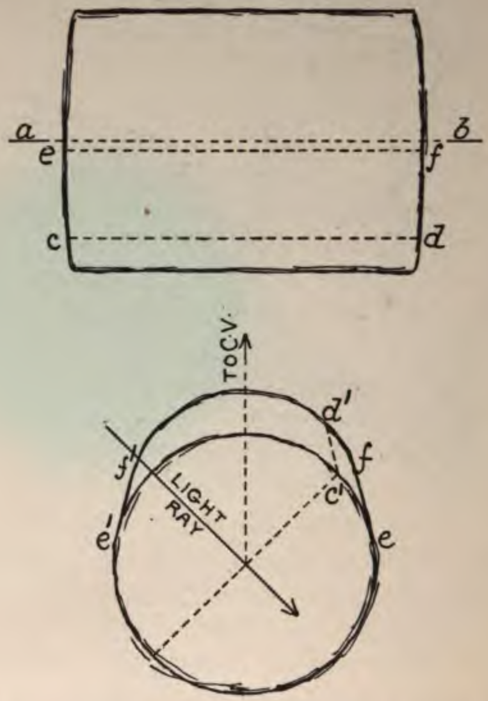


Fig. 105 is a cylinder lying on a horizontal plane in a fixed position on one of its elements, with axis, *ab*, parallel to the picture-plane. If the light-rays make an angle of 45° with the horizontal plane, what will be the various positions of the line of shade as they are assumed to fall first parallel to the picture-plane, then at increasing angles until they fall at an angle of 45° to both the horizontal plane and the picture plane? that is, a vertical plane passed through a ray of light would be perpendicular to the picture-plane. In the first a tangent plane of light would touch the cylinder along an element that was just

FIG. 105.



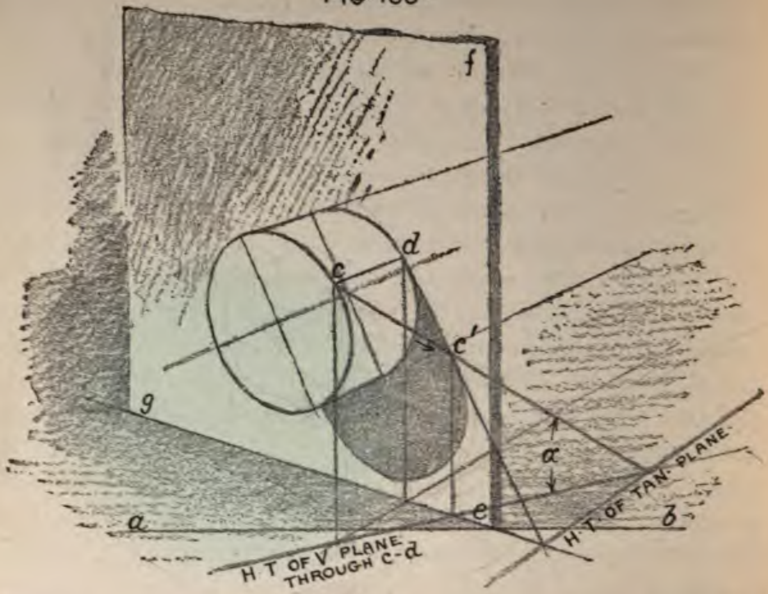
in the middle; according to the elevation of the eye and the distance of the observer, the picture of this element would be coincident with or slightly removed from the cylinder axis; in any usual case it would not be far from the middle, as *ef*, if the eye were above the axis. As the light source is moved through greater angles to the picture-plane this line of shade or element of contact would move downward on the cylinder until it took the position, say, of *cd*, the exact location of which is not important but could be obtained by projection, similar allowance being made for the elevation of the eye above the cylinder, as was done at *ef*. The extremes of position are the elements *ef* and *cd*, and as the light-rays move through equal angles the line of shade will move through equal proportional distances between these two lines.

Assume now that the direction of the light-rays is fixed in any one position, say in the last one mentioned above. If the cylinder move from a position in which its axis is parallel to the picture-plane to one in which it is perpendicular, what changes will take place in the position of its line of shade? In the first this line would be as at *cd*, and when the cylinder had moved to the second extreme position it would be as at *ef* and *e'f'* in the lower cylinder of the figure. These changes are similar to those before observed. Assuming the light-rays parallel to the picture-plane, with a similar rotation of the cylinder, *ef* of the upper cylinder would be the first position of the line of shade, and *c'd'* that when the axis of the cylinder was perpendicular to the picture-plane, found by erecting a perpendicular to the light-rays through the center of the base of the cylinder. As in the first case mentioned, a proportional variation in the position of the line of shade accompanies a given change in the position of the cylinder, and the limits in each case are not far apart. Hence it is seen that by a comprehension of the nature of these changes a little reflection will enable one to place approximately correct the line of shade on any cylinder whose axis is horizontal, and with the light-rays taken anywhere between the limits mentioned.

154. The Cast Shadow of a Cylinder on the Plane of its Base.—It has been mentioned that if the element of deepest shadow upon a cylinder were at once assumed it would not be easy to cast its shadow without ascertaining by construction just what the direction of light-rays would be to produce the assumed conditions, unless the shadow were to be cast on a plane coincident with or parallel to the base. It is now proposed to investigate this last case, which is quite a common one in practice.

Fig. 106 is a cylinder extending out from, and its axis perpendicular to, a plane surface *efg*. Assume the element of deepest shadow, as *cd*. Since this element is theoretically derived by a tangent plane of light, the trace of this plane or shadow of *cd* on the plane *efg* will be a tangent to the base of the cylinder; let *c'* be the shadow of *c* assumed arbitrarily. What conditions as to direction of light-rays have been assumed? *cc'*, *c'd*, and *dc* are all lines within the tangent plane. The horizontal trace of the line *c'd* is in the line *eg*, i.e., at the intersection of the plane of the base of the cylinder with the horizontal plane. If through this horizontal trace a line is sketched parallel to the cylinder axis, that is, parallel perspectively to *cd*, then where *cc'*, prolonged, cuts this line will be *its* horizontal trace. The angle which the light-ray *cc'* makes with the horizontal plane is found by passing a vertical plane

FIG. 106.

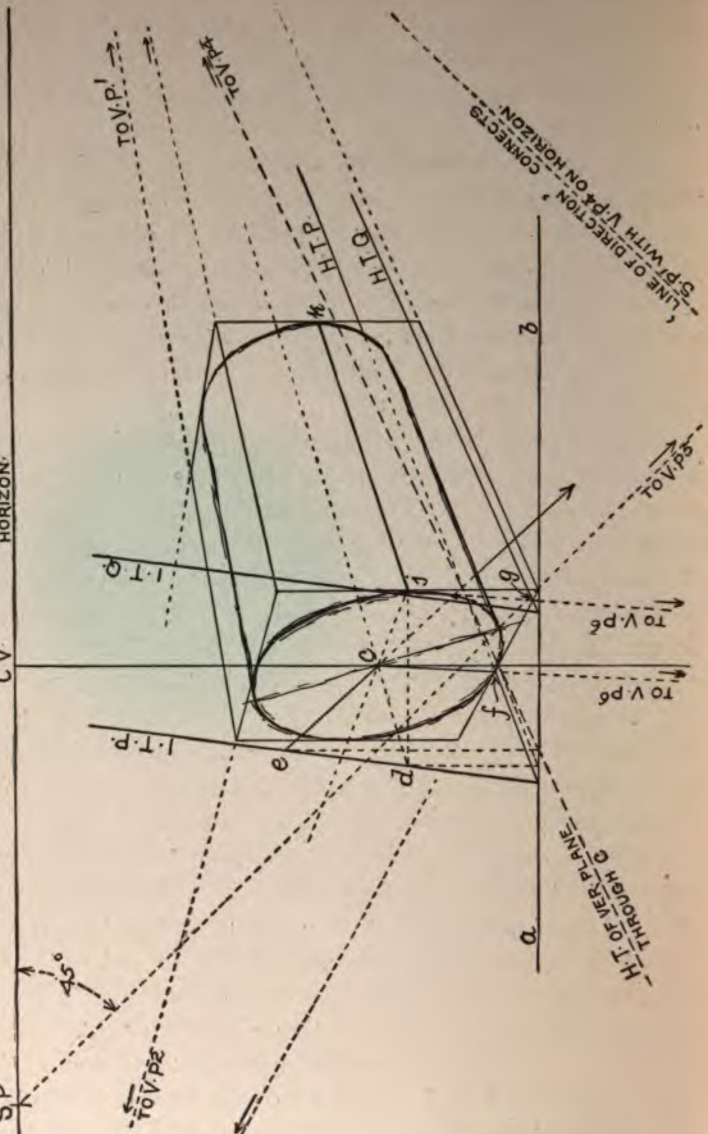


through it. The horizontal trace of the light-ray is one point in the horizontal trace of this vertical plane, and the horizontal projection of *c′* in the line *eg* is another point. Hence *α* is the picture of the angle which the light-rays make with the horizontal plane.

155. Linear Perspective Construction of the Line of Shade on a Cylinder.—It is instructive to know just how the line of shade may be accurately placed on a cylinder by linear perspective; hence it is here proposed to discuss briefly this more rigid process, one, by the way, which can be extended with slight modifications to apply to any ordinary curved surface having straight-line elements. It is an application of the principle given in section 153.

Fig. 107 is a cylinder lying on one of its elements on the horizontal plane, these elements vanishing in $V.P.^1$, and the plane of the bases in a vertical line through $V.P.^2$ The rays of light are assumed to fall at an angle of 45° to the horizontal plane, and a vertical plane which may be drawn containing them also stands at the same angle to the picture-plane and its vanishing-trace goes through $V.P.^4$ Following the plan described in section 153, the initial and horizontal traces are first drawn of a plane *P* which passes through the cylinder axis and contains a ray of light. The initial point of the cylinder axis is *d*, and is one point in the initial trace. A light-ray through any point *c* of the axis is next drawn; by means of its horizontal projection which is in the plane of the base of the cylinder and the vertical plane passed through the light-rays, its initial point is found to be *e*, which is a second point in the desired initial trace; the horizontal trace, since the plane contains the axis of the cylinder, must vanish in $V.P.^1$ In order to get the traces of a parallel and tangent plane to the cylinder, a line *cf* is next drawn; it is the line of intersection of *P*, the plane just derived, and the plane of the base of the cylinder. Its initial point would be at the intersection of the initial traces of *P* and the base of the cylinder, and its vanishing-point on the vanishing-trace of the latter at $V.P.^6$ A tangent to the cylin-

FIG 107.


S.P'''
45°
HORIZON
C V
TO V.P.1
TO V.P4
TO V.P2
TO V.P3
TO V.p6
TO V.p6
H.T.P.
H.T.Q.
I.T.P.
I.T.Q.
a
b
c
d
e
f
g
h
j
LINE OF DIRECTION CONNECTS S.P.III WITH V.P4 ON HORIZON
H.T. OF VER. PLANE THROUGH C

der, *jg*, is now drawn parallel to *cf*, which by its intersection with the horizontal plane in *g* gives one point in the horizontal trace of a tangent plane of light, *Q*; the initial trace of the latter is parallel to *P*. *jk* is the element of deepest shadow.

156. Cast Shadows upon a Cylindrical Surface. — In general a straight line will cast a shadow on a cylindrical surface which appears as a curve, and this curve is derived by getting the shadows of points as before described; except where the plane of shadow is easily constructed, auxiliary planes must be passed through a number of points of the line each cutting the cylinder in a curve. The shadows of the several points will be found on these respective curves of intersection with the cylindrical surface. To simplify the work involved, the various cutting planes are assumed as parallel and, as before suggested, are so chosen that, if possible, they cut out straight lines from the cylinder. Fig. 108 is a simple case of this, a cylinder with horizontal axis inclined to the picture-plane. To the left is a vertical stick which is near enough to cast its shadow on the cylinder. The horizontal trace of the invisible plane of shadow of the stick is assumed to be the line *de*. The intersection of this plane with the surface of the cylinder, which of course will be the required shadow, is very easily derived by sketching a series of intersecting planes which are parallel to the axis of the cylinder; the intersections of these planes with the cylinder will be straight lines. Take one plane for example through the cylinder axis: its intersection with the cylinder will be the lines *i*,*g*,*k*,*h*, its intersection with the invisible shadow of the stick will be a vertical line, *lj*; hence, as can be seen by following the construction, *j* will be one point in the curve of the visible shadow. As many points can be found by the same process as are desired. If the stick is short enough to cast the shadow of its extremity upon the cylinder, then this point will be where a ray of light through the extremity cuts the curve of intersection with the cylinder of the invisible shadow; in the figure this point happens to be coincident with *j*.

Before leaving the example illustrated in Fig. 108, several deductions from it are worth noticing. The line of shadow of the stick is an elliptical curve when the invisible plane of its shadow is oblique to the axis of the cylinder, and is sharpest in curvature nearest the stick; since the cylinder is horizontal and the invisible shadow plane is vertical, the ellipse giving the form of the intersection in this case is parallel to the picture-plane and has a horizontal major axis, which would help to determine by inspection the probable shape of the shadow. The major axis of the curve would, moreover, be longer if the plane of the invisible shadow were more oblique to the cylinder axis, i.e., a condition which arises if the cylinder were to be turned about horizontally, or the light source changed with respect to the picture-plane. Again, a fact is instanced here which has before been stated, that as a surface recedes from an object casting a shadow upon it, the shadow recedes from the object;

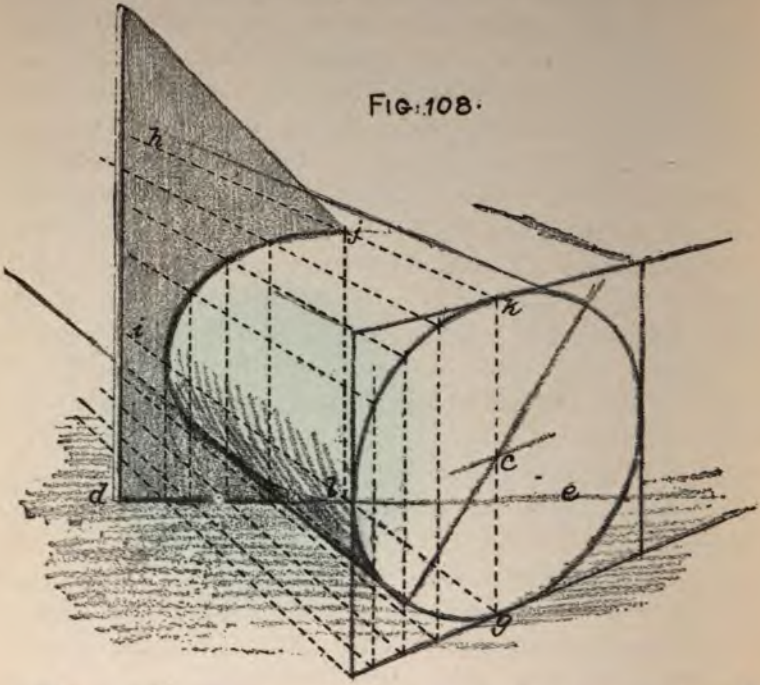
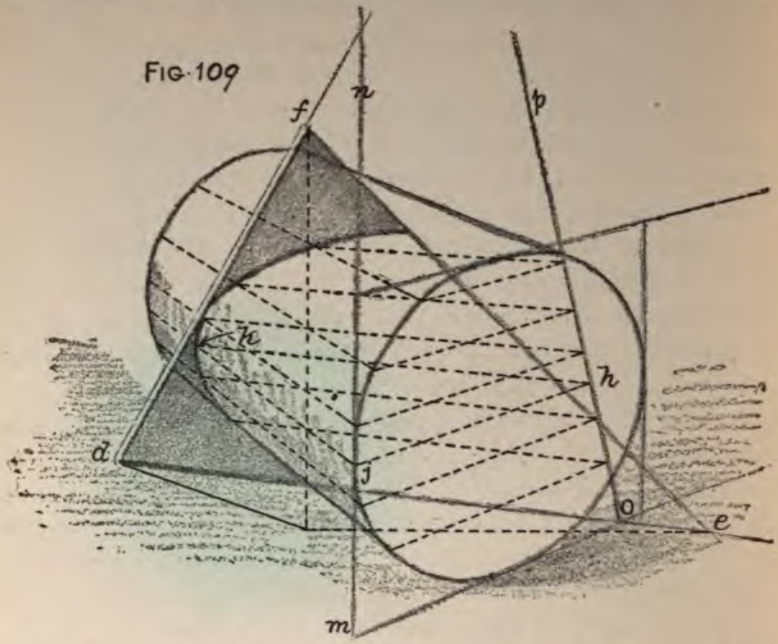


FIG. 108.

this, were the line bent or broken, would be a key to the probable shadow form. However, while it may be true that the shadow recedes from the object as the receiving surface recedes from it, when we are dealing with curved forms, another important consideration comes in which can be best illustrated in the case of a bolt with a cylindrical head. Here each point of that portion of the curve of the head which casts its shadow on the bolt is equally distant radially from the surface of the bolt, while the line of shadow will be the curve of intersection of two cylinders. If a modification in the interpretation of the law is made and it is understood that the distances described are *distances between points and surfaces measured along lines in the direction of the rays of light*, then it would be of general application. The above is a law which is very helpful indeed in approximating shadows without construction; cast shadows are frequently not represented at all sharply in pictures, and therefore only a rough approximation to the exact form is required.

Fig. 109 illustrates a very instructive problem in cast shadows upon a cylinder, which is, however, not so easy to derive as the case just described; it is that of the shadow of a line which is oblique to the horizontal plane and to the axis of the cylinder. Several methods may be used, one of the simplest of which will now be described. Horizontal planes are passed to cut the cylinder in straight lines and also the invisible plane of shadow of the line in horizontal lines. *def* is the invisible plane of shadow of the line *df* when the light-rays are parallel to the picture plane, as can be seen by following the construction; the plane of the base of the cylinder cuts the picture-plane, let us say, in the line *mn* and the plane of the invisible shadow of the stick in the line *op*, found by connecting the horizontal traces of the cylinder base and the plane of shadow of the line with their initial traces; the construction is not fully carried out. Any one point in the curve of cast shadow may be found by considering, for example, the auxiliary horizontal plane *jhk*. *jh* is a line cut out of the plane of the base of the cylinder and will be parallel, perspectively, to *mo*; *hk* is a line cut out of the invisible plane of shadow of the stick and will be perspectively parallel to the line *de*. *k*, a point in the shadow, will be where the latter line intersects *jk*, an element cut from the surface of the cylinder. Other points in the shadow may be similarly found.

FIG. 109



157. Surface and Cast Shadows of Cones.—The right circular cone may be considered as a special case of the cylinder, in which the elements meet at a finite instead of an infinite distance; problems, therefore, in the surface and cast shadows of the cone are solved in the same manner as those of a cylinder, due account being taken of this difference. Planes to cut straight lines from the cone, for example, must be passed through the apex of the cone.

The general rule for finding the line of shade on a conical surface may be stated as follows: *Pass a ray of light through the apex of the cone to cut the plane of the base produced; from* *this point draw tangents to the curve of the base; lines connecting these points of tangency with the apex of the cone will constitute a part of the line of shade.* The tangents to the curve of the base, it may be observed, will form the limits of the shadow of the cone on the plane of its base, if the light source is on the same side of the base as the apex. If a ray of light through the apex falls within the base of the cone, no part of the conical surface is theoretically in shadow; this can be instanced in the heads of conical rivets, which under general lighting do not have surface shadow, since the axis of the conical head is very short as compared with the diameter of the base.

The form of the cast shadow of a circular cone on a plane surface is in general a plane bounded by an elliptical curve and two intersecting tangents to it.

It can be noticed by an observation of these surfaces in general that much less surface shadow is visible than would exist on a cylinder similarly placed and similarly lighted. The sloping of the surface towards a point is the cause of this, and, moreover, all contrasts between tones are sharper near the apex than they are near the base, since there are fewer intermediate tones. The cone also presents the phenomena of reflected light, which last is most pronounced furthest from the apex.

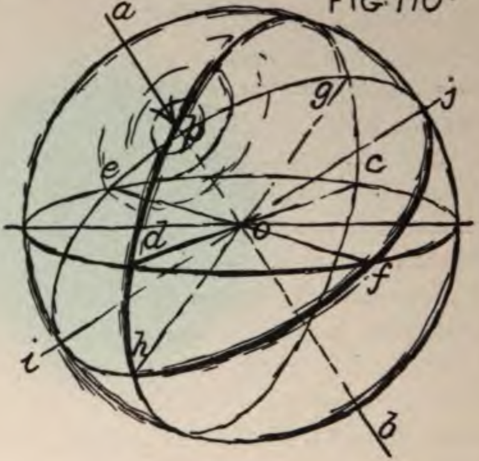
158. Surface and Cast Shadow of a Sphere.—The sphere affords probably the most interesting case of shadow phenomena that we have, as well as the most difficult problem to represent satisfactorily, though the theory of its shadow can be made very simple.

The contour of the surface shadow of the sphere will, under all circumstances, be a circle, seen generally as an ellipse, and can be thought of as formed by the line of tangency of a cylinder of rays of light whose axis is a diameter of the sphere; and further, the invisible shadow of the sphere being a circular cylinder, its intersection with a plane surface, causing the visible shadow, will be either an ellipse or a circle, more generally the former in perspective. The least shadow of the sphere on a plane would be when the plane was perpendicular to the light-rays.

The general method of ascertaining constructively the line of shade on a sphere may be stated as follows: *Pass a ray of light through the center of the sphere; through the center also pass a plane perpendicular to this ray; the line of intersection of this plane with the sphere will be the required line of shade.*

If the approximate directed plane method is used in applying this rule, the constructive work is rendered very simple.

FIG. 110.



See Fig. 110. *ab* is a ray of light assumed to fall at the angle *aod* to the horizontal plane *fced*; its intersection with the sphere, which would locate the point *p*, approximately the region of high light, is found by sketching the vertical plane *cgdh*; a line *og* perpendicular to the light-ray in the plane *cgdh*, and another perpendicular to it in the plane of *fced*, namely, *of*, give two lines in the required plane perpendicular to the light-ray. Its curve of intersection with the sphere goes through the points *fgeh*, and its major axis, perpendicular to *ab*, is *ij*. If the shadow of the sphere were desired on some plane surface, shadows could be cast of a number of points in this curve by drawing a series of parallel lines from it normal —preferably—to the receiving surface; then the triangular planes of shadow of these construction lines could be completed by the method previously described, and the shadows of their extremities preserved as points in the shadow line of the sphere. Or again, the shadow may be roughly approximated by getting the shadow of the center of the sphere only, and finishing by sketch method.

The difficult problem in the shadow of the sphere is in the proper grading of it by means of any medium used in a drawing, and it is rarely that a completely satisfactory result is obtained; the blending of tones cannot be carried too far or else flatness in the effect is the result, yet there must be sufficient gradation to give an impression of a spherical surface. If the physical principle that "the angle of incidence is equal to the angle of reflection" be ignored, then the high light on the sphere is a small circular region as in Fig. 110, with center at *p*, and the regions of equal intensity of illumination on the sphere will be theoretically on circles whose centers are severally on the line of the light-rays *ab*; hence the increase in the tones of the shading on the sphere may be said to progress by means of a series of narrow circular bands down to the region of deepest shadow. It is impossible to represent these in a drawing in great enough numbers, and it is equally impossible for the eye to distinguish them on any given spherical surface, but all artistic shading of such is made on this assumption, and as many of these bands interpreted and then blended, one with the other, as is deemed necessary to give the proper effect of the form. The above treatment presupposes of course that there are no exterior and modifying conditions, that the sphere is free from the influence of other bodies.

CHAPTER XI.

PERSPECTIVE SKETCHES FROM WORKING DRAWINGS.

159. General Statement.—In section 87 reference was made to a profitable application of perspective principles by making sketches from working drawings. The working drawing is something which represents the facts concerning the form of a subject by a collection of views, each of which deals with a certain class of those facts. It is the technical man's shorthand, his way of expressing himself more briefly than would be possible by written or spoken words. In making sketches from working drawings, the student must keep this clearly in mind, that the several views together are necessary to represent *one* thing, and that when sketched perspectively all the facts are collected and incorporated in the sketch result. This can be exemplified by the drawings given of the simple problems at the end of Chapter VIII. In such sketching as this, experience and the application of perspective principles are the only assistance available. Experience leads to the storing of knowledge as to the appearance of forms under various conditions, and the application of perspective principles helps to classify experience and to give true impressions when experience fails. The principles of perspective should be used as heretofore indicated in the problems referred to above, not as limiting conditions, but as helps.

160. General Method of Application.—The first requisite for beginning drawing of this kind is to have a thorough knowledge of the form to be drawn, as represented in the working drawing. After this has been acquired the sketch can be intelligently begun. The perspective conditions must next be analyzed. The principal edges of most subjects to be drawn, mechanisms, architectural features, etc., can be grouped into three systems, the three which are exemplified in the edges of a rectangular solid, such as a square prism or a cube. If there are other directions to edges of the forms, they can be referred to one or the other of these three systems in terms of their angularity to them. Moreover, it generally occurs that one system will be that of vertical lines, so that only two are left in which perspective foreshortening or convergence needs consideration; these should be located in the sketch by taking, for convenience and accuracy, the outside lines of the systems; if the proper convergence is given to the extreme lines, those parallel and lying in between are more readily located. Locating principal perspective systems in this way induces a comprehensive method of treatment, in harmony with general artistic methods, and it will be found very necessary in dealing with complicated forms to have a system which will lead expeditiously to accurate results. Sketching from a model proceeds by an observation of exterior form and direction of edges and by an estimation of proportional values, while sketching from the working drawing must be largely constructive, the exterior form of a subject being arrived at, it may be said, deductively. Imaginary axial lines are sometimes helpful, sometimes imaginary cutting and tangent planes, and a quite certain dependence upon the fundamental principles of perspective.

In applying perspective principles two general methods or a combination of the two may be employed advantageously: (1) the circumscribing or inscribing figure method; (2) the axial method. These will be treated in detail under separate headings.

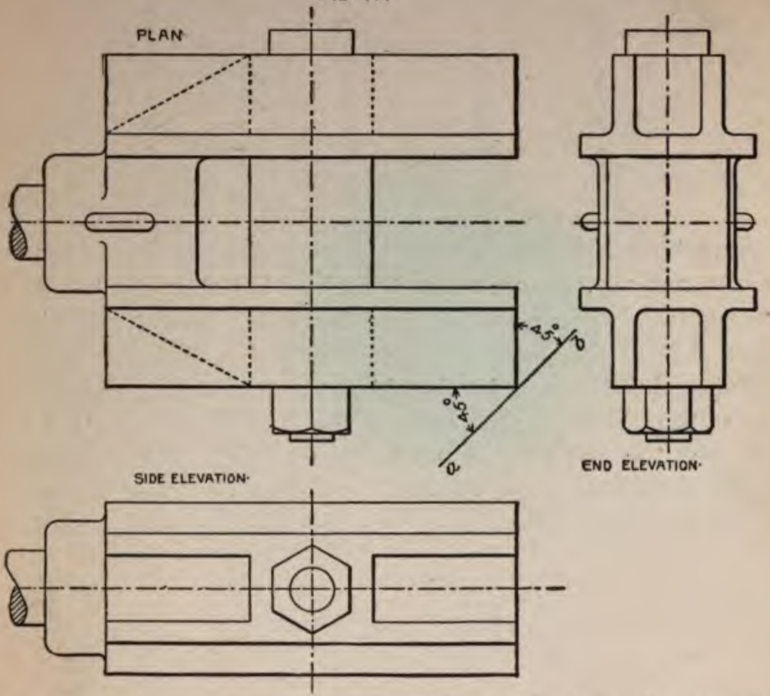
161. Construction by Circumscribing or Inscribed Figures.—If a subject consists mainly of one rectangular solid, it will be found most convenient to deal with the perspective of that one form, using a unit cube, by which to estimate its relative foreshortening of faces and edges, and then make, by adding or taking away from this form, the necessary changes in order to give it the final result. If it consists of a combination of rectangular solids quite compactly arranged, then it may be well to divide the consideration of the perspective features into those of each of the several forms involved. The cube as a unit may again be used to estimate position and foreshortening. These separate rectangular solid forms may not at first be apparent in a subject, and in fact they are many times quite arbitrary divisions, to facilitate the interpretation, and sometimes again made independently of the forms involved. The choice of the division into the solids is one part of the problem involving judgment. The general principle underlying the division may be stated as that which will reduce the perspective principles to the simplest basis and leave the least changes to be figured out in which the final form differs from the type solids.

The slight modifications, spoken of above, in which the final forms differ from the type solids chosen will probably in many cases be forms in which the edges are oblique to those of the type forms. In such cases it is well to remember that, where estimating the value of an angle, the plane of the angle and the axis of the plane are the first things to locate. If the plane of the angle is in or parallel to one of the faces of a rectangular solid used, its axis will be parallel to the edges which are perpendicular to that face.

The choice of a point of view is a desirable thing to decide upon carefully, so that those features which are characteristic, and which it is purposed to show, will, when put into perspective, appear clear; for this purpose if the plan of the subject is taken and a line drawn just touching it representing the plan view of a picture-plane, it will indicate, by slight inspection, about how much of the various parts of the form will show if the angles of the edges or vertical faces of the solid are assumed to make the angles with the picture-plane which they do with this line touching the plan. Such a line will be used in the examples to follow and is uniformly lettered *ab*.

162. Illustrative Examples.—Fig. 111 shows a plan, a side, and an end view of a simply constructed cross-head for a steam-engine. It changes the straight-line reciprocating action of the piston—shown broken at the left—into the rotary motion of the fly-wheel by the aid of an intervening connecting-rod which takes hold of the pin shown fastened in, in the center. The piston-rod lies horizontal normally, hence the side view is partly that of vertical surfaces, and the plan that of horizontal surfaces.

FIG. 111.



To simplify the problem of sketching it, assume, in the plan, that the picture-plane would occupy the position relative to it of the line *ab*. This it can be seen will cause the cylindrical portions to become almost hidden and only straight-line construction to remain. A circumscribing construction is advisable which takes in the entire subject, excluding the nut on the bolt; this is shown laid out in Fig. 112. The angles of the vertical faces with the picture-plane are equal, foreshortening of edges equal, and the proportions of the whole based on a cube; one face is twice as long as high, and the other slightly over twice. The axis of the piston-rod coincides with a diameter of the solid, and it is found by sketching the intersecting diagonals as shown; likewise the center of the large face is located as the center of the base of the hexagonal nut. The remaining lines to be drawn are parallel, with few exceptions, to these three systems of lines. Fig. 113 shows the completed outline sketch; the farther cylindrical end of the cross-head shows slightly, its curve being so short, however, that it hardly deserves analytical notice. Constructive methods in such cases will be described later. The hexagonal nut is first laid out,

FIG 112

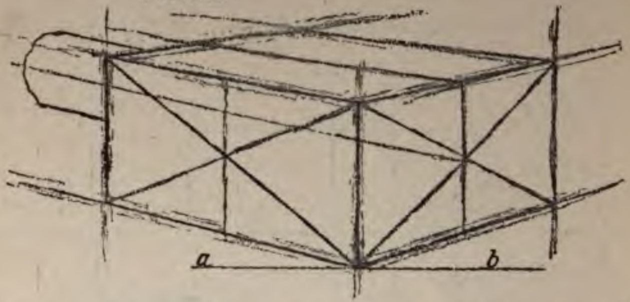
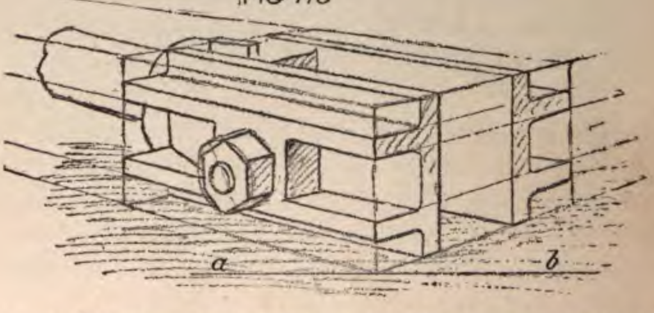


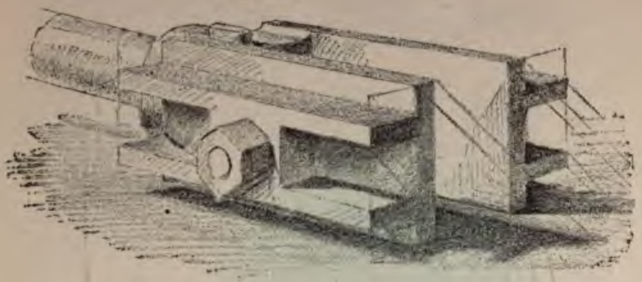
FIG 113



for convenience, by means of a circumscribing cylinder, made the proportions of the nut; the bolt shows as an elliptical curve whose major axis is perpendicular to the axis of the bolt.

Fig. 114 is a completely shaded drawing of the same subject; the light-rays are assumed to make an angle of 45°

FIG. 114.



with the horizontal plane and to be parallel to the picture-plane. All the principal points of the shadows may be obtained by sketching the 45° triangles, as shown in a few cases.

In this and the succeeding examples discussed, the analysis into constructive methods has been more or less minute, more so than perhaps is necessary, but it has been done in order to demonstrate the fact that perspective construction can be used advantageously where other means fail. The more detailed construction can and should be dispensed with wherever it is possible to do so without the result suffering by it.

Fig. 115 is a set of views of a pillow-block for shafting, mounted on a support; it is somewhat more difficult than the preceding and involves an analysis into several fundamental forms. Fig. 116 shows the fundamental layout. The base is first sketched at the apparent inclination which its edges are to the picture-plane as shown by the line *ab* in the plan. Since the short edge makes the least angle, it is taken as a unit of measures, and a horizontal square is used to estimate the probable length of the long edges. The center of the base is next found, and, by the construction easily followed in the drawing, the trapezoidal support is laid out, the short edge still

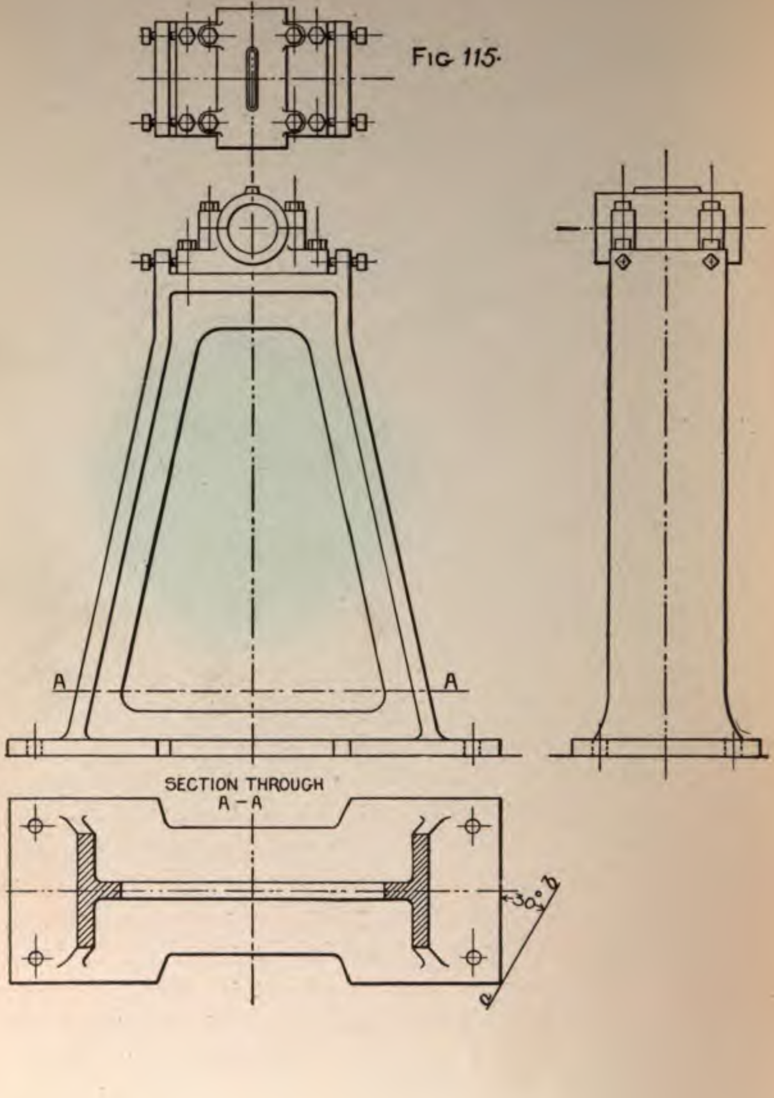
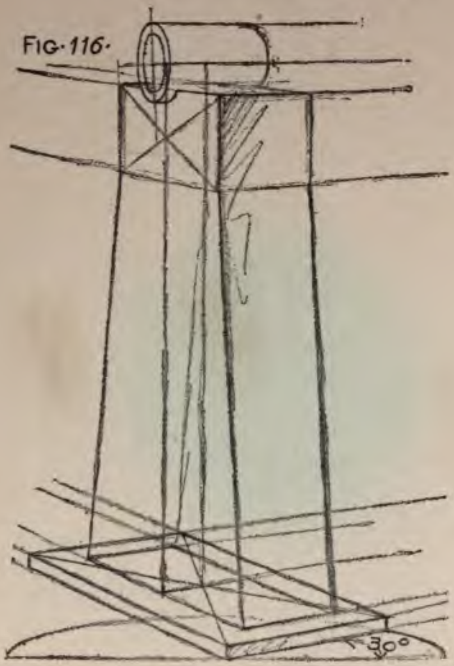


FIG. 115.

serving as a unit of measures; near the top the form is changed to a rectangular solid mounted on top of the trapezoidal one. The cylinder of the bearing, it is to be noted, is longer across

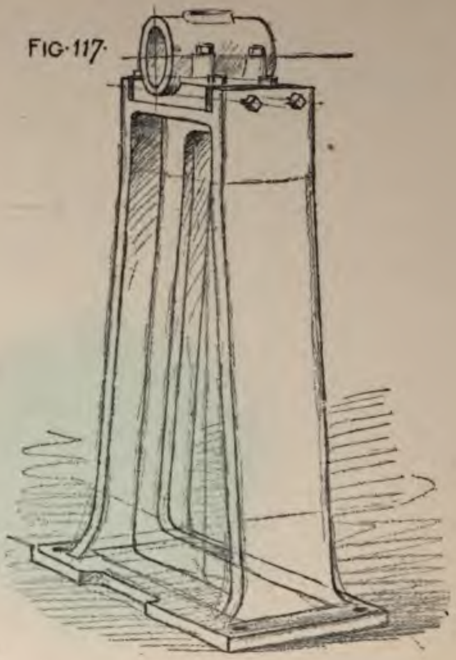
FIG. 116.



than the base of which it is a part; this difference is shown as estimated along the axis of the cylinder. The proper foreshortening of the circular end can be estimated if need be in the manner heretofore indicated for the drawing of circles. Fig. 117 shows the completed outline sketch; most of the constructive lines have been preserved to show how the slight modifications of the fundamental form are made. It is in such subjects as these that care has to be observed in the proper treatment of circular forms; the rule given in Chapter V, section 65, should be remembered and followed.

Fig. 118 is a subject forming an interesting application of the method of circumscribing figures. It shows a plan and two end elevations of a unit of a marine-engine bed. The principal parts are of the form of channel-iron, but can be treated

FIG. 117.



in sketching by considering them as long rectangular solids. Fig. 119 shows the finished outline; the angles with the picture-plane of the receding vertical faces were taken as slightly unequal, that is, one slightly greater than 45° and the adjacent one slightly less. It was not thought necessary to show the preliminary layout by an additional figure; it is preserved, however, conventionally by dotted lines in the final drawing. After constructing the rectangular solids spoken of above, the additional work involved trimming these away to show merely flanges connected by a central web, and the raised parts added the required vertical distances from the upper plane of the four main arms. If this subject were shaded, the light should be assumed to come from such a direction as would light all the visible vertical surfaces and throw shadows in the various recesses; light-rays which are assumed such that a vertical

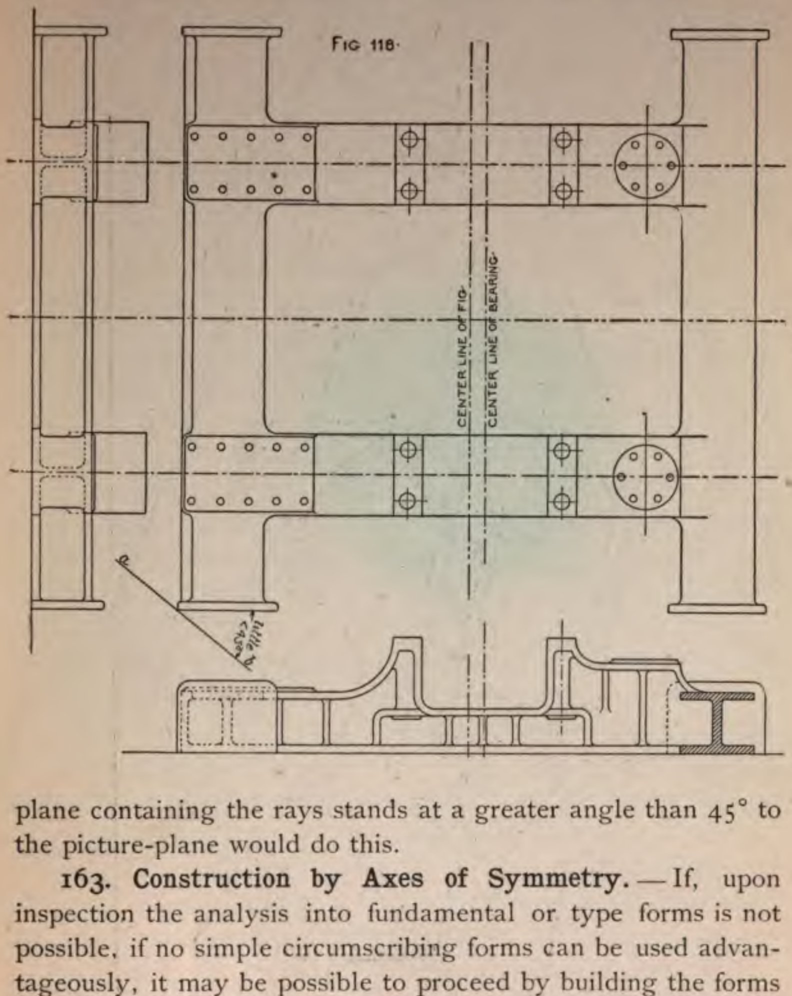


FIG. 118.

plane containing the rays stands at a greater angle than 45° to the picture-plane would do this.

163. Construction by Axes of Symmetry.—If, upon inspection the analysis into fundamental or type forms is not possible, if no simple circumscribing forms can be used advantageously, it may be possible to proceed by building the forms up about an axis or axes of symmetry; this method is particularly applicable where a series of cylindrical bodies is involved. It deserves to be noticed that if forms are thus referable to axes of symmetry, the distances between them, if they be separated, are most conveniently measured along or parallel to these

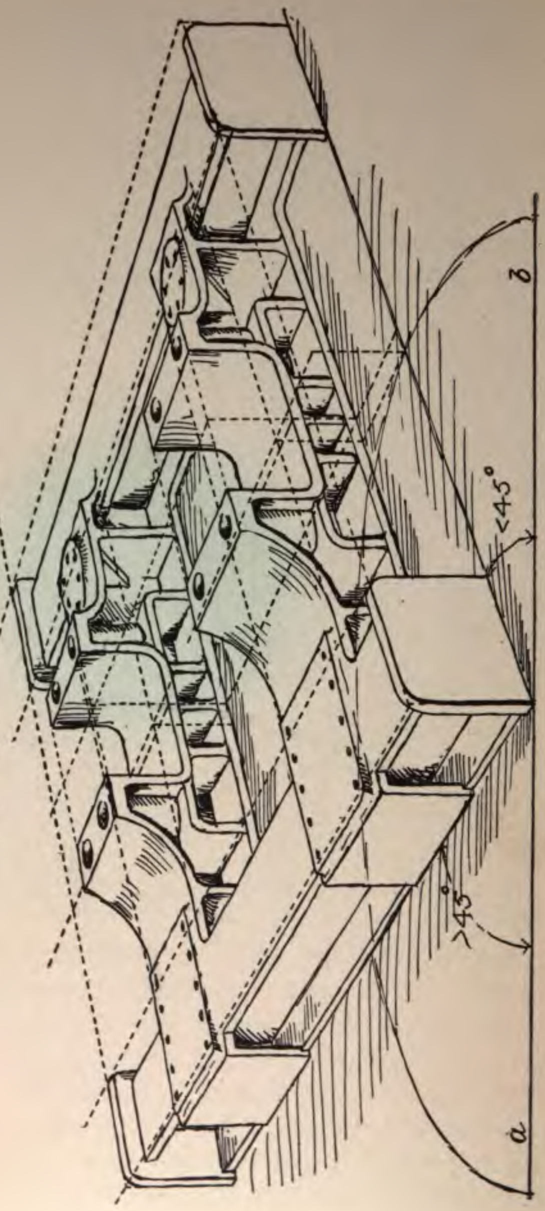
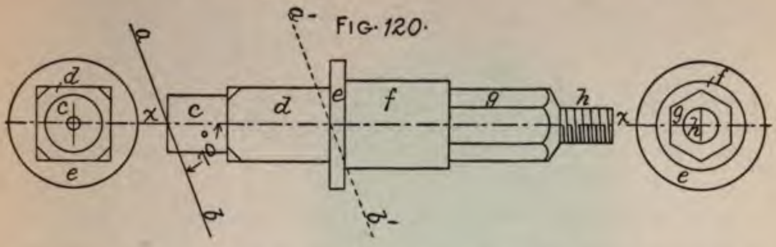


FIG. 119.

axes, otherwise measurements being taken oblique to the axes will be found very difficult to estimate at all accurately, involving as they would different conditions of foreshortening and angularity to the other systems of lines. If this method of construction is kept clearly in mind, then neither axial construction, nor construction by circumscribing figures ought to present special difficulties; either can be used as preferred.

164. Illustrative Examples.—Fig. 120 is a side and two end views of a rock-shaft, a very simple illustration of axial

FIG. 120.



construction. Any direction can be given to the axis; it was assumed as horizontal, and in order to illustrate forcibly the fact mentioned in the preceding paragraph, namely, that the distances between parallel planes should be computed along their common axis, this axis was placed at a large angle, 70°, to the picture-plane. Fig. 121 shows the skeleton layout. Since the collar in the center is the largest feature, it was placed first; its value was obtained by constructing a horizontal directed plane, which also served to scale the relative foreshortening of other measurements parallel to the axis, such as *e*, *f*, and *g*. These distances between the various plane surfaces perpendicular to the axis are shown in the side elevation as *c*, *d*, *e*, *f*, *g*, and *h*, and their foreshortened values by the same letters in the sketch, all estimated with a unit which is that portion of the axis lying within and limited by the directed plane. Observe carefully, in such subjects as this, that, given a series of parallel plane surfaces, mounted upon an axis which is oblique to the picture-plane, those further removed from it will show a relatively larger amount of their surfaces; this applied to circular planes means that the further circular planes have a larger ratio of minor to major axes. Fig. 122 shows the complete outline drawing of this subject.

FIG 121.

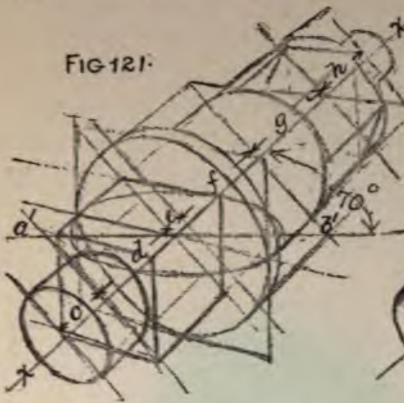
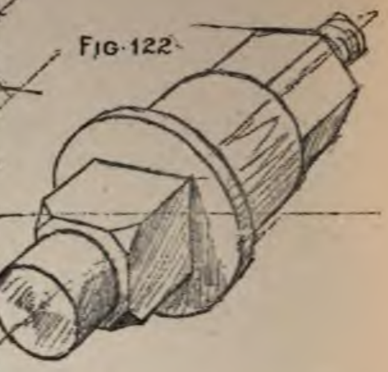


FIG 122.



Another case of axial construction is shown in Fig. 123, which is a side and an end elevation of an overhung crank; it has two principal parallel axes, *x*, the axis of the shaft, and *y*, the axis of the crank-pin; a third axis is used to connect the two, namely, *z*, the axis of symmetry of the plane face of the crank-arm. Fig. 124 shows the preliminary blocking out. The choice of position was such that the axis of the shaft makes an angle of 45° with the picture-plane and stands horizontal; this will give the same angle for the plane of the circular ends. The first part laid out was the front face of the crank, the foreshortening of the cylindrical ends being obtained as before by the directed plane method; the taper on the arm was effected by noting the points of contact of the front edges with the circular ends shown by the dotted lines *cc'* and *dd'*, being lines perpendicular to the cylinder axis and horizontal. Fig. 125 shows how the remaining features were added, the only additional construction being that which will give the correct apparent distances between the various circular planes. Fig. 126 is added to show the complete effect when construction lines are erased and the whole shaded. The direction for the light-rays was not at first assumed, but instead the element of

FIG 123.

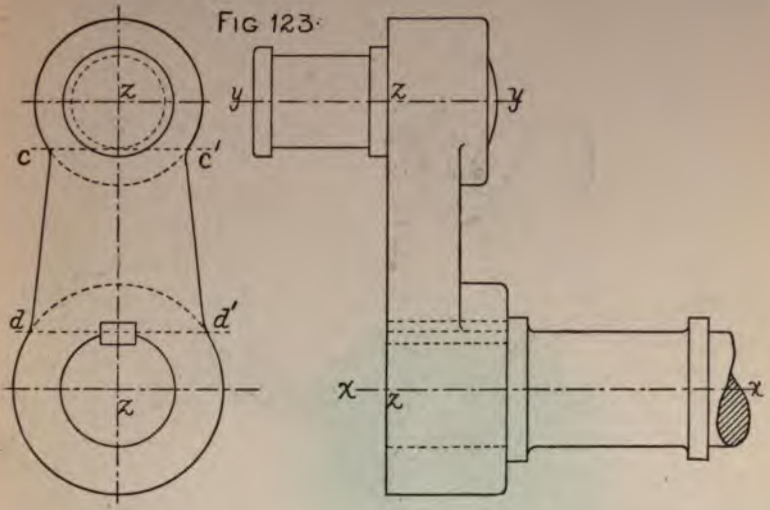


FIG 124.

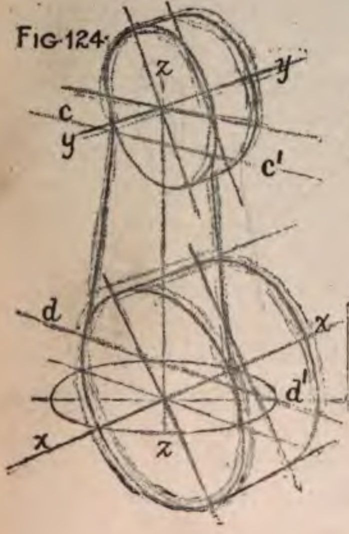
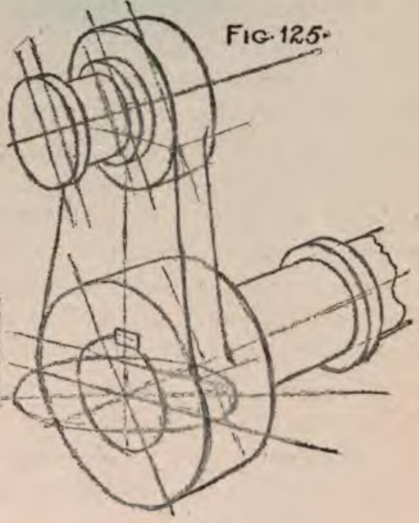


FIG 125.



deepest shadow, so that there would be insured sufficient light and shade contrast on the subject; from the region of deepest shadow, that of the high light was obtained by the construction shown on the hub of the crank, by radial arrows.

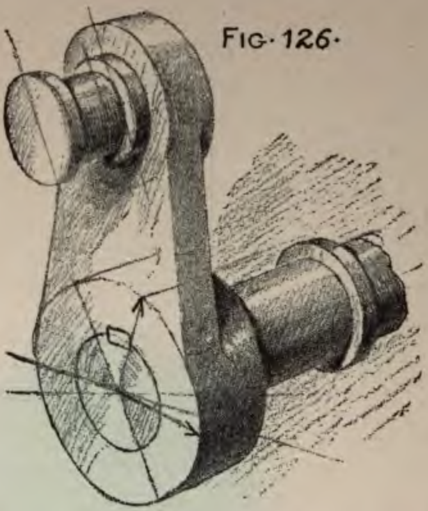


FIG. 126.

165. Combination of Circumscribing and Axial Construction.—Certain forms met with may have several axes of symmetry, but these axes may not be all equally prominent, or it may be that some circumscribing construction can be used to better advantage than by building up about an axis; in other words, it may many times be found expedient to combine simple rectangular forms with axes of symmetry and lay out the sketch by using both in such a combination as will require least construction. It will generally be found that in subjects at all complex this combination of methods will be necessary in order to simplify the work.

In locating axes of symmetry connected with the simple forms as described, it will very likely be found expedient also to make use of cutting planes passing through the faces of the type forms and containing the axes.

Irregular curved forms are apt to give considerable trouble at first. Careful analysis, however, will lead to one of two general processes, viz., referring the lines or surfaces to rectangular forms and estimating perspectively in how much they differ from the straight edges of these familiar type solids, or, second, to refer the lines or surfaces to circles or cylinders or a combination of two or more circles or cylinders in which the radii are different, the centers being located by axial construction, that is, by sketching the axes of the base forms used; this will be illustrated in the examples to follow.

166. Illustrative Examples.—Fig. 127 shows a side elevation and plan of a form of connecting-rod, such as is used to connect the piston through the cross-head to the crank. The right-hand end is the cross-head end, and the cross-head pin is shown in the fork of the rod. The rod is of considerable length, but shown broken and the ends drawn close together. In analyzing the form it is seen that x, y, and z are axes of symmetry, but ABC is a simple rectangular form which it would be easier to deal with first hand than to locate the axis x and about it circumscribe this form. In the case of the cross-head end a circumscribing rectangular solid might similarly be drawn around the entire fork, but the modifications which would have to be made from this form make it advisable to build up the pin and the fork about the axis y and draw it as a cylinder and connecting loop. Fig. 128 shows the fundamental layout made in the above manner, the angles with the picture-plane being those shown by the line ab in the plan and in the layout. The rectangular solid ABC was first sketched, its proportions being obtained by using a horizontal directed plane of a radius equal to one-half of the long edge of the solid, which is also its depth; the axes x, z, and y, respectively, were next obtained, y crossing z at any convenient point, since the connecting-rod has been broken away. The cylinder of the forked end was then built up, its proportions being arrived at by prolonging lines from the rectangle at the left, which were one unit apart, thus getting the value of that unit in the neighborhood of the fork; in addition the value of the faces of the

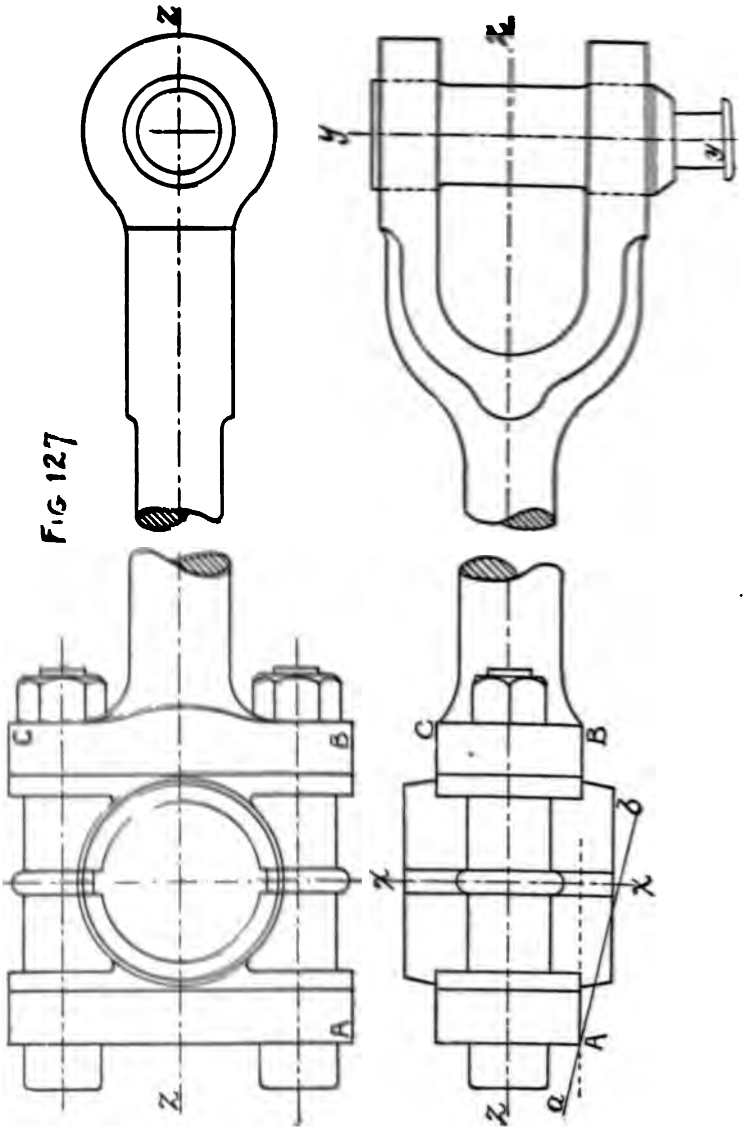
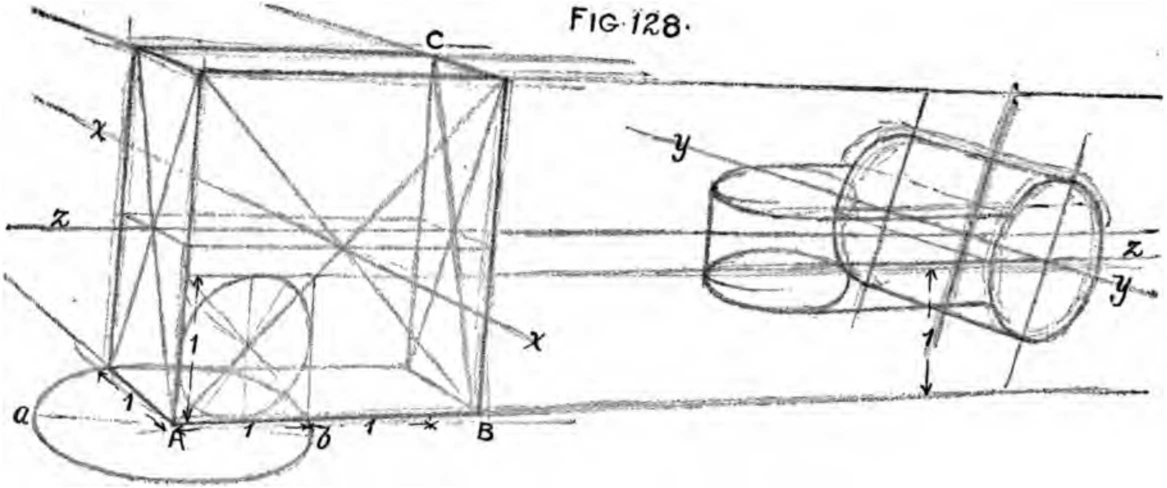


Fig 127

FIG·128·



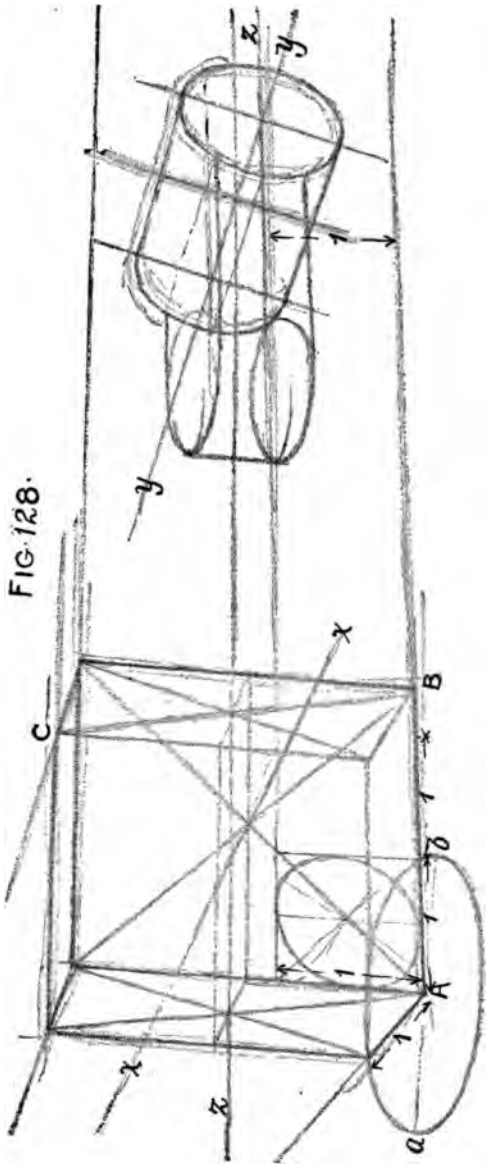


FIG. 128.

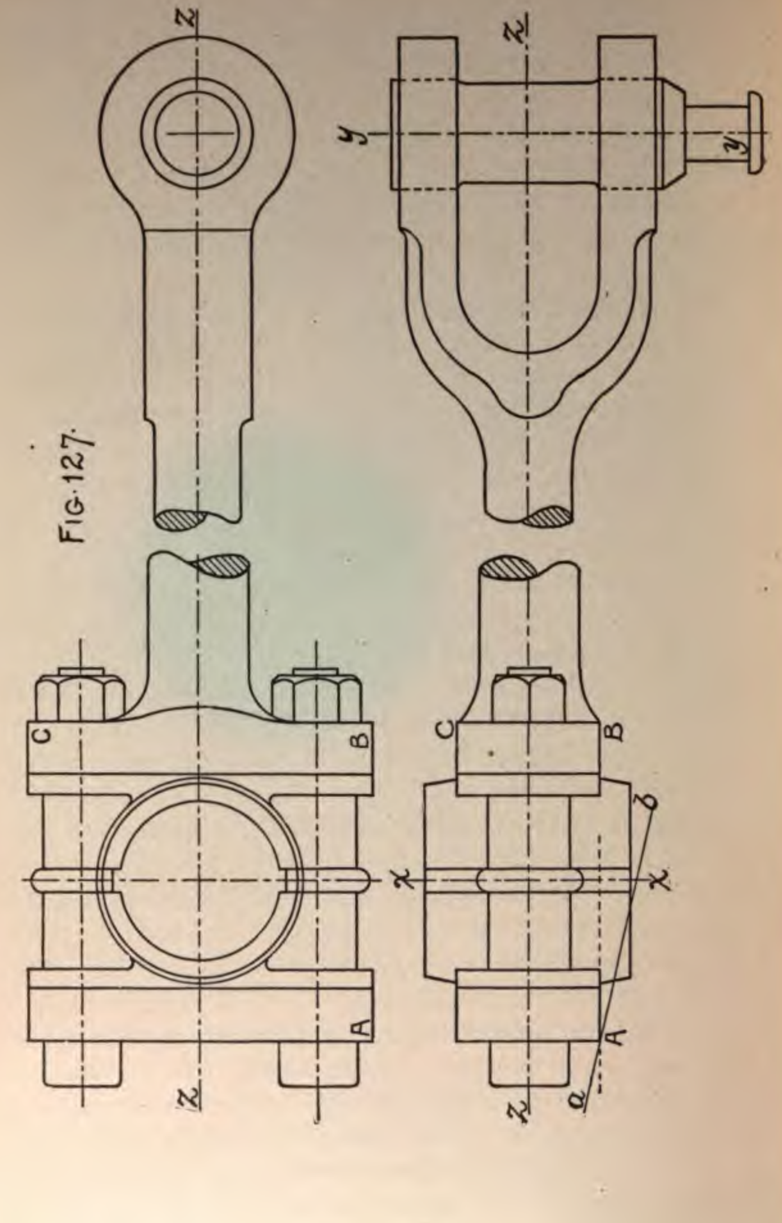
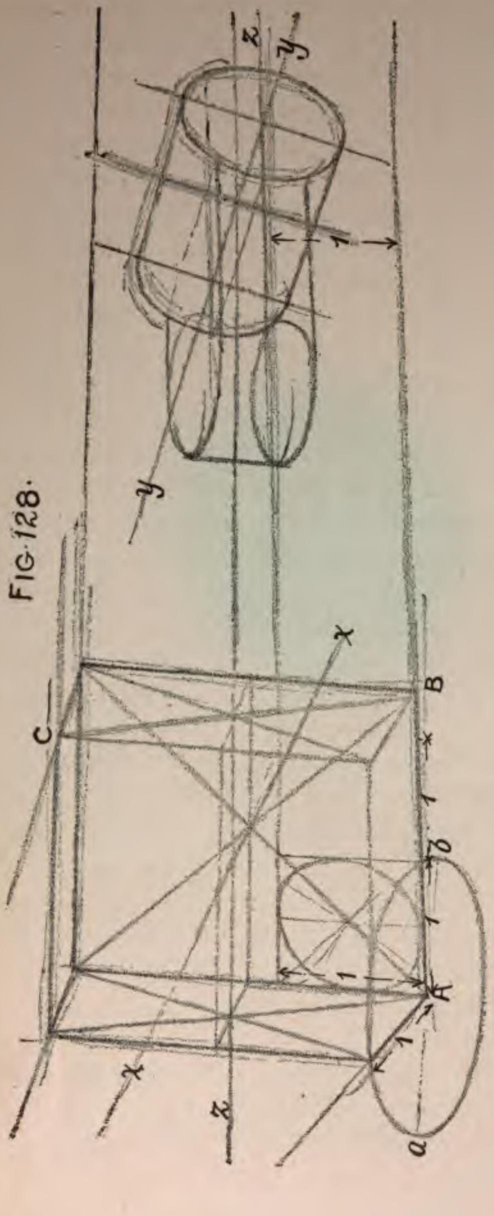


FIG. 127.

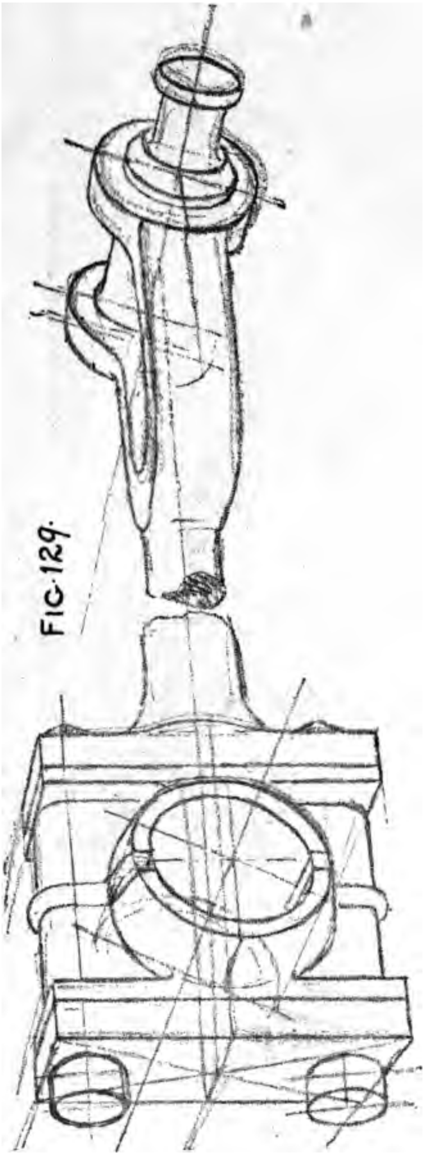
FIG. 128.



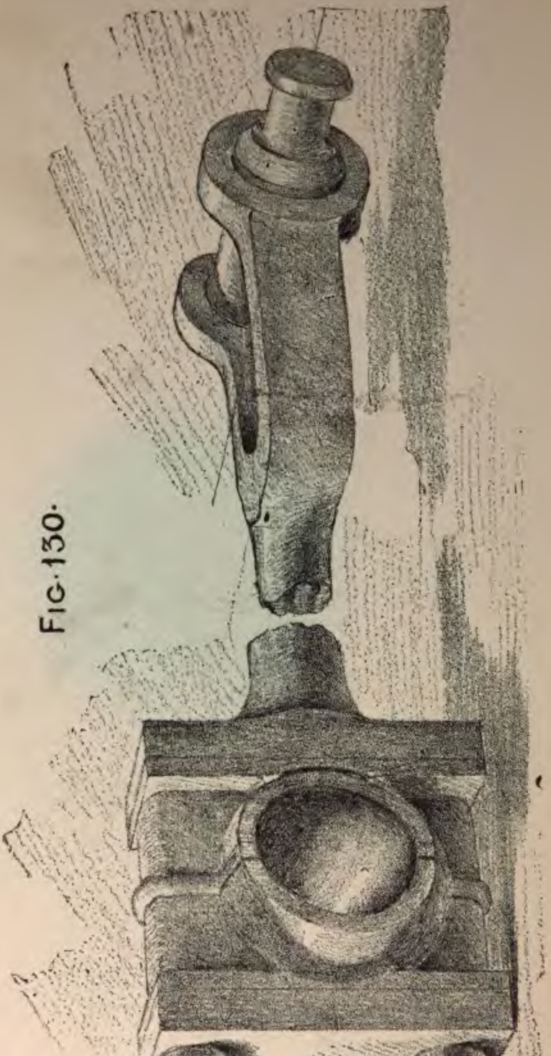
cylinder were obtained by sketching a square on the face of the rectangle as shown and from it obtaining the ratio of axes of the ellipse required to represent the near base of the cylinder. The planes of the two ellipses are parallel and, owing to their relative positions with respect to the picture-plane, that on the axis y will have a slightly smaller ratio than the other. Fig. 129 shows the completed outline, the construction lines being shown which were required in obtaining the additional modifications. These constructions should be sufficiently clear from an examination of the drawing not to need further explanation, except that attention deserves to be called to the careful construction necessary to ascertain whether the further curve of the opening in the bearing-pieces shows, and just where it would come if it did show. The distances between the planes of the front and back curve will be of course the distances between the large faces of the rectangular solid plus twice the distance the plane of one curve is from a face of the solid. Fig. 130 is a completely shaded drawing of the same with the light-rays assumed parallel to the picture-plane.

No particular construction was necessary for the cast shadows further than has already been discussed in the preceding chapter.

The case of a form which is largely handled by circumscribing construction is that shown in Fig. 131, which is a plan, a side and an end elevation, together with a sectional detail, of a bracket supporting a platform on a large engine. A rectangular solid $ABCEF$, which circumscribes the entire form, is used first. Its length is slightly less than twice its height. Fig. 132 shows the complete sketch. The dot-and-dash lines give the position taken by the preliminary sketch lines. To get the curve of the bracket it is possible to use an arc of a circle which approximates closely the curve, using O as the center of it, and through its perspective sketching for reference, the major axis of the ellipse representing the circle. An additional approximation might have been used if it were thought necessary by which to complete the curve, but this is not shown in the

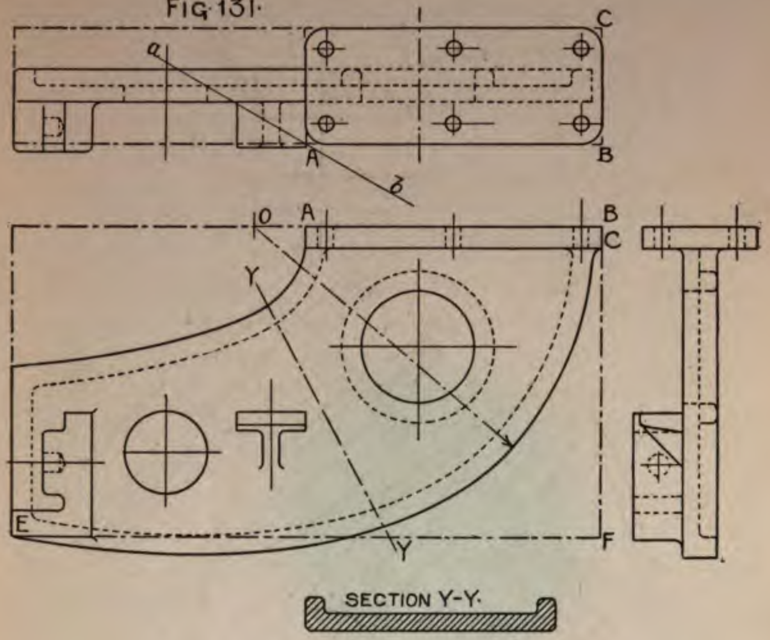


FIG·129·

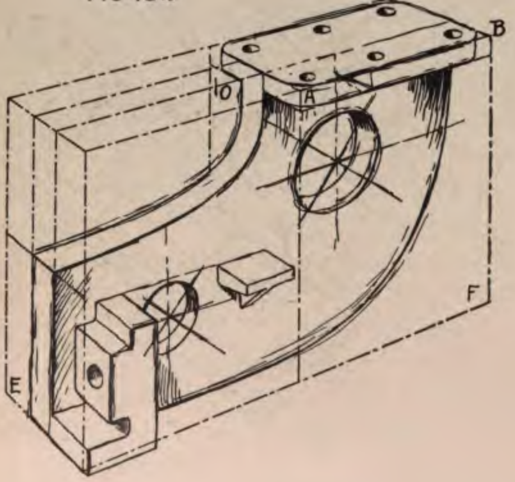


FIG·130·

FIG·131·



FIG·132·



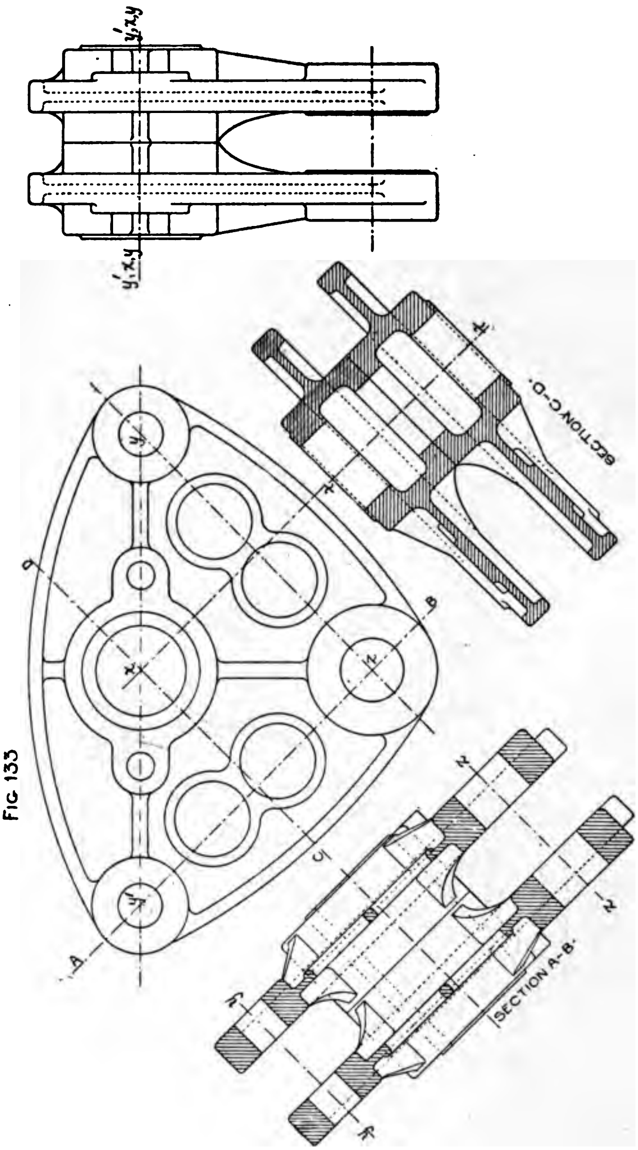


Fig 133

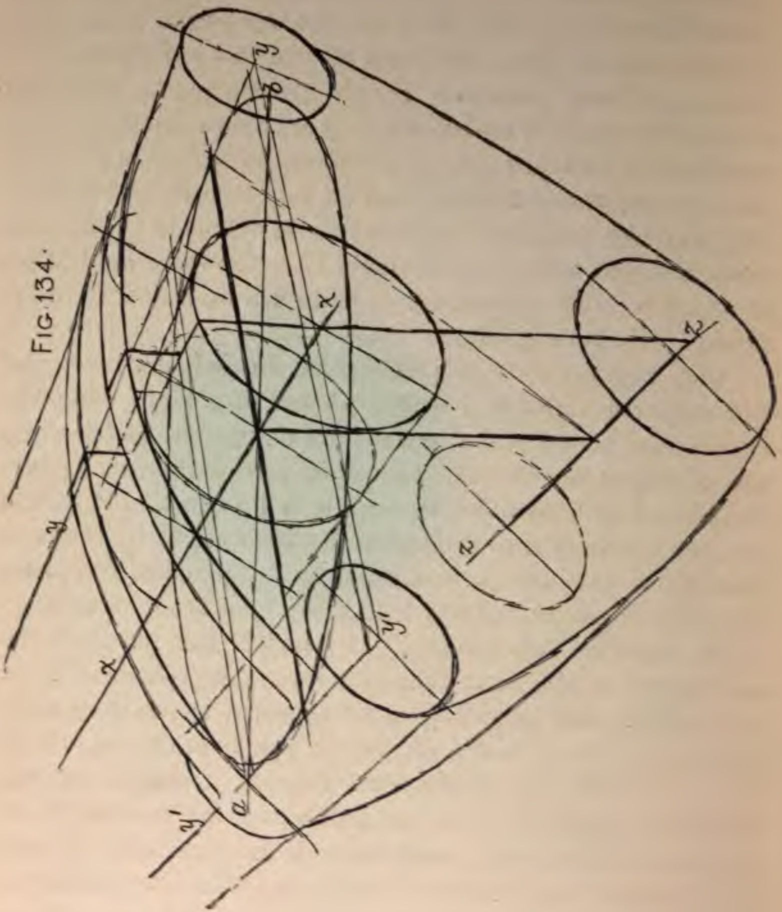
sketch. The remainder of the work hardly needs further comment.

It is now proposed to discuss briefly several examples of sketching considerably more intricate than the preceding. Before doing this it is well to observe by way of preparation that too implicit faith should not be placed upon the absolute accuracy of any particular sketch construction. Although constructive methods are helpful, they are approximations which may have to be adjusted to fit the forms as desired in the later development of the drawing, and in very complicated subjects anything like analytical construction in details is as impracticable in sketch methods as it would be in the more rigid linear perspective. The adjustment of parts will be instanced in the examples that are to follow.

Fig. 133 shows a side elevation, an end elevation, and two sectional views of a walking-beam of a large pumping-engine; the planes of the sections are designated by letters on the corresponding views. The main part of the casting is a plate of metal with holes through it to lighten it, and several portions enlarged into cylindrical bodies to form bearing surfaces whose axes are perpendicular to the plane of the plate; the whole is surrounded and held stiffly by a heavy flange.

In sketching this subject it is probable that the constructive features are the most difficult parts of the problem. There are to begin with several axes to consider, shown in part in the sectional views, and projected as points in the side elevation. x is the axis of the main bearing, z that of the one vertically underneath it, a line connecting the two, or, better still, that vertical plane passed through and containing the two, being a plane of symmetry of the form. The side bearings to the right and left of x have axes y and y' parallel to x, and the plane which may be passed through all three is in the drawing a horizontal plane. Fig. 134 is the fundamental layout for the sketch; the four axes contained within the two planes described were first located.

In order to get the desired position of the subject two



FIG·134·

alternatives presented themselves: (1) To locate the horizontal plane containing the axes y, y', and x. This plane may be considered to have as its bounding edges the axes y and y' and lines connecting their extremities, i.e., lines lying within the planes of the faces of the upper right- and left-hand cylindrical bearings. (2) To locate the three axes severally by taking x and an auxiliary axis, which is a line perpendicular to x at its center and terminated by the axes y and y'. This last was chosen as the best. A directed plane was used to estimate the inclinations of the axes to the picture-plane, the radius of which was the distance between the axes x and either y or y', and its center at the intersection of x and the auxiliary axis. The angles were not assumed any particular value in degrees, but one as considerably smaller than the other, so that the principal face of the walking-beam has but slight inclination to the picture-plane. Next the axis z was located its proper vertical distance below x, measured from the center of the directed plane and estimated by reference to the major axis of that plane. The distance of y and y' from x was also estimated from the foreshortened radius of the directed plane, that is, by the auxiliary axis, cutting x through the center. The proper convergence of the several axes was approximated by judgment based upon the relative values of the angles chosen. Assistance can be obtained for locating y and y' from the fact that they will be lines tangent to the curve of the directed plane.

Next the centers of the faces of the several cylindrical bearings, mounted upon the axes, were obtained; these it will be noticed are not all in the same vertical plane, but have to be separately located either side of a vertical plane of symmetry of the figure; this vertical plane of symmetry is one which is perpendicular to the one previously mentioned, and is a plane which contains the centers of the axes of all the cylindrical bearings. After locating the respective centers, the ellipses representing the bases of the cylinders were drawn their proper relative size, the minor axes coincident in each case with the cylinder axis. At this point it is well to reflect, before going further, upon what will probably show of the completed subject and what will remain hidden; construction which will ultimately be hidden it is perhaps possible to ignore. Fig. 135 shows the completed outline drawing. It was developed from the fundamental layout in the following stages: The outside flanges *c*, *d*, *e*, *f*, *g* were sketched, connecting the outside bearings, and the curvatures of each roughly estimated; they could have been arrived at more accurately, if desired, by locating the center of the approximate circular arc taken from the side elevation.

Next the junction lines of the main plate with the bearings were sketched in, and the strengthening webs drawn; lastly the holes through the plate were located together with the small finishing touches. There is considerable convergence of the axes of the bearings towards a vanishing-point not very far distant on the paper, so that the ellipses sketched about them do not have their respective axes parallel. This may perhaps be considered a blemish upon the good appearance of the subject, and it is probable that it would strike most observers in this way; to overcome this a slight adjustment of each curve in the direction of parallelism of axes can be made, —not a complete compensation, but a step in that direction. Assuming a greater distance of observer, i.e., making the vanishing-points of the receding horizontal lines come further apart, would have accomplished the same result. The above compensation was not made in the figure lest a misunderstanding of the constructive methods might result. Fig. 136 is added as a suggestion of how this may be accomplished, and also how the whole may be shaded.

Fig. 137 shows several views of a cap to the discharge-chambers of a large pumping-engine, namely, a plan, a sectional elevation longitudinally through the center on the axis *x*, and two transverse sections, designated by capital letters. The whole is a somewhat irregular form and affords an excellent example for the application of analytical methods which

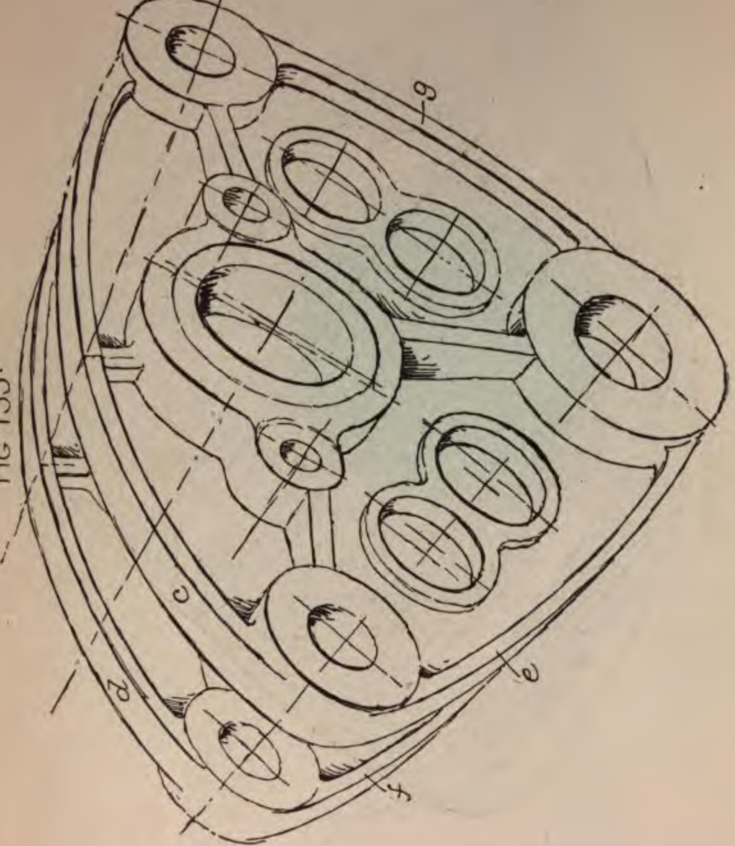


FIG 135.

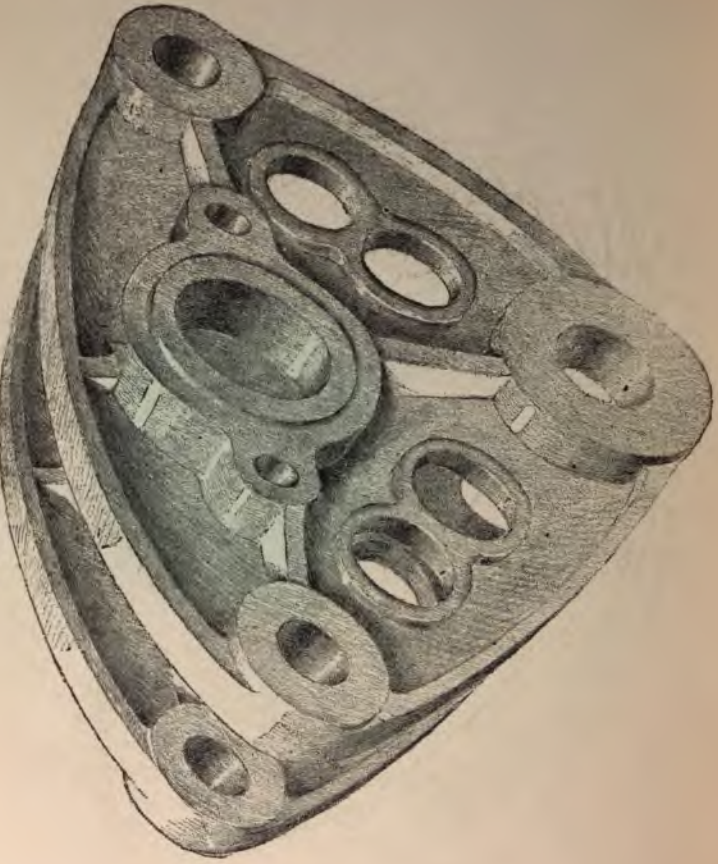


FIG. 136.

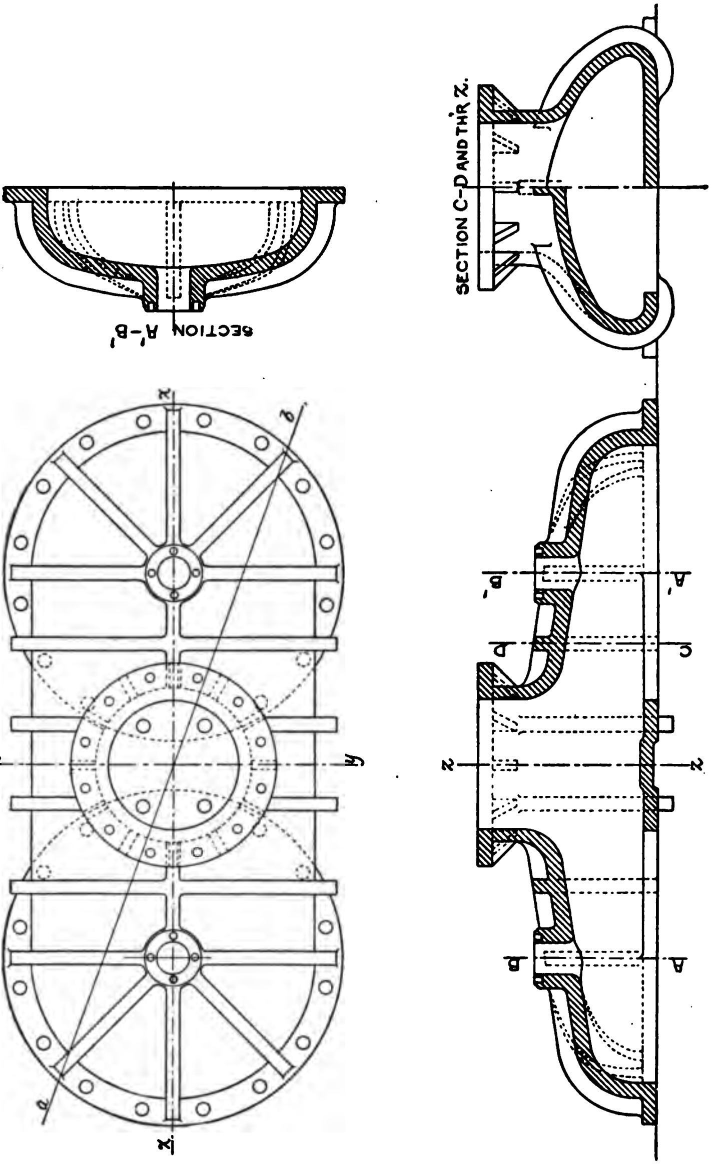
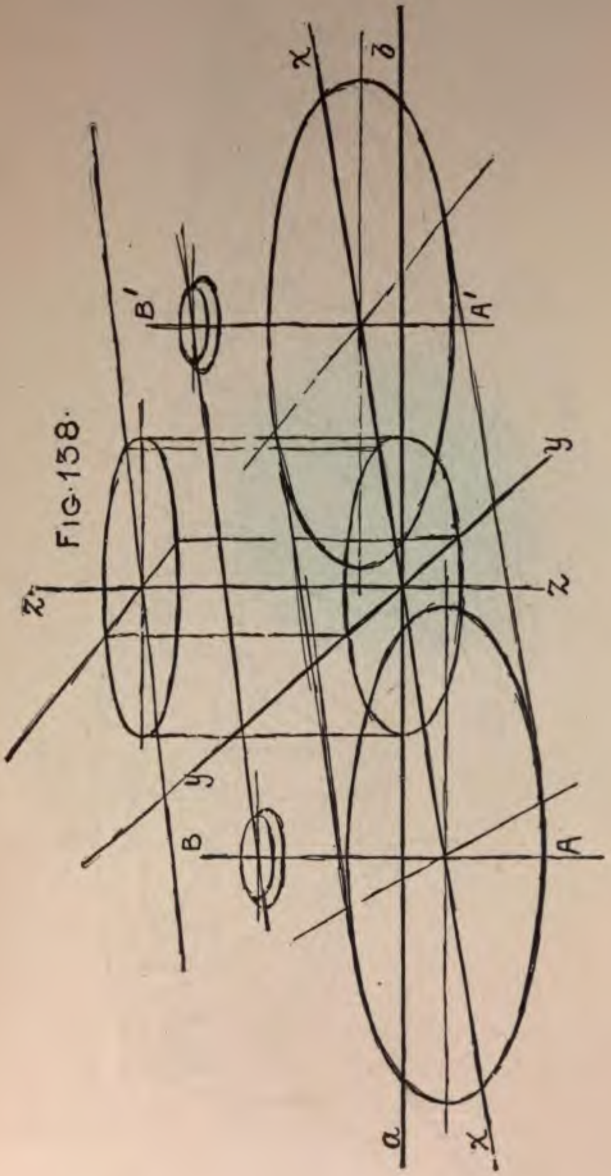


Fig. 137

will make the treatment simple. It differs from the preceding case in that the constructive processes available are few in number, while the finishing work is the most difficult. The dome can be represented in its curvature only by a correct drawing of the lines of intersection of the various webs which pass over it; constructive methods for the drawing of these last while possible are not profitable, and a "cut-and-try" process by judgment of effect is the safest. Fig. 138 gives the fundamental layout on a larger scale than the working drawings. The lower plane of the flanges of the base received first consideration; the line connecting the centers of these was inclined to the picture-plane at the angle shown by the reference-line *ab* in the plan of Fig. 137. Placing the reference-line through the center of the plan in this way would result, if the same scale were used as in the working drawings, in an increase of the actual scale in working towards the observer and a decrease the other way, parts of the final form being slightly larger than they are in the actual drawing; this of course could be compensated for if desired. The upper plane of the upper flange was next drawn, first in the lower plane just located and then projected up the required vertical distance. The upper plane of the right- and left-hand openings was then placed with reference to these. This constitutes the sum of the fundamental construction that was used. Fig. 139 shows the complete outline and embodies in visible lines any additional detailed construction that was used. In this, as in the preceding example, much is hidden and need not be figured out. The central longitudinal rib was the first to be sketched, then the transverse one over the center of each flange; the first decides practically how much of the subject will be obscured, and so at once limits the region of visible parts. From the final result it is at once seen how judgment must go hand in hand with the perspective construction and guide in getting a good effect. In more complicated subjects detailed analytical methods will fail entirely because the construction required to effect them gives opportunities for errors to a greater extent than would probably

FIG·138·



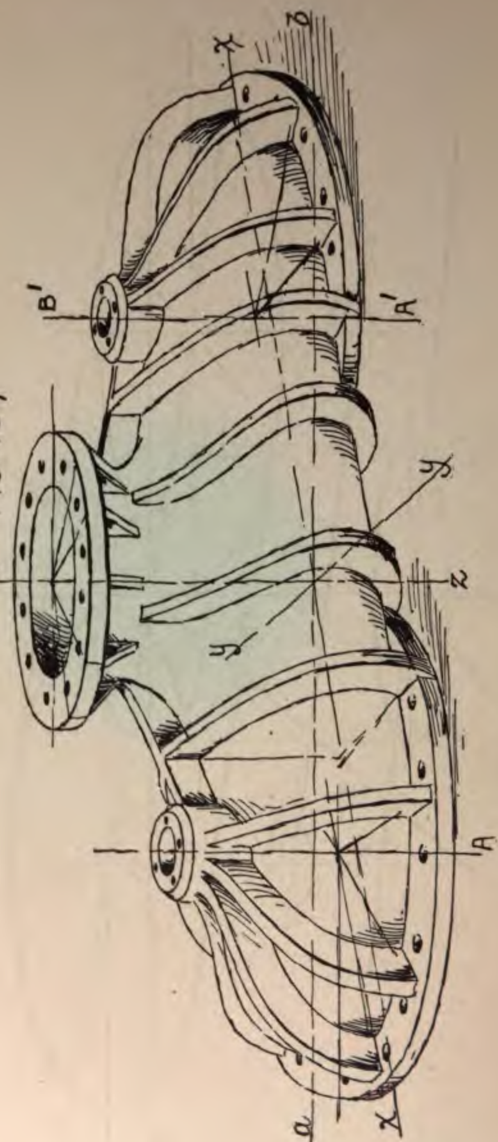


FIG. 139

arise were the judgment of the effect of the result wholly relied upon. With constant practice, the application of perspective principles becomes unconscious, just as do the motions of the hand to form a given letter in writing or to combine the proper letters to spell a word, while the intellect is left free to concentrate its efforts upon the shaping of ideas expressed by the words.

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